James M Cook

AN

INTRODUCTION

TO

ALGEBRA,

WITH

NOTES AND OBSERVATIONS;

DESIGNED FOR THE

USE OF SCHOOLS AND OTHER PLACES OF PUBLIC EDUCATION.

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THE NINETEENTH EDITION.

CORRECTED AND GREATLY IMPROVED.

TO WHICH IS ALSO ADDED,

An Appendix, containing a Synopsis on Variable Quantities.

By SAMUEL MAYNARD.

——Ingenuas didicisse fideliter artes
Emollit mores, nec sinit esse feros.—OVID.

LONDON:

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PREFACE.

The powers of the mind, like those of the body, are increased by frequent exertion; application and industry supply the place of genius and invention; and even the creative faculty itself may be strengthened and improved by use and perseverance. Uncultivated nature is uniformly rude and imbecile, it being by imitation alone that we at first acquire knowledge, and the means of extending its bounds. A just and perfect acquaintance with the simple elements of science is a necessary step towards our future progress and advancement; and this, assisted by laborious investigation and habitual inquiry, will constantly lead to eminence and perfection.

Books of rudiments, therefore, concisely written, well digested, and methodically arranged, are treasures of inestimable value; and too many attempts cannot be made to render them perfect and complete. When the first principles of any art or science are firmly fixed and rooted in the mind, their application soon becomes easy, pleasant and obvious: the understanding is delighted and enlarged; we conceive clearly, reason distinctly, and form just and satisfactory conclusions. But, on the contrary, when the mind, instead of reposing on the stability of truth and received principles, is wandering in doubt and uncertainty, our ideas will necessarily be confused and obscure; and every step we take must be attended with fresh difficulties and endless perplexity.

That the grounds, or fundamental parts, of every science are dull and unentertaining, is a complaint universally made, and a truth not to be denied; but then, what is obtained with difficulty is usually remembered with ease; and what is purchased with pain is often possessed with pleasure. The seeds of knowledge are sown in every soil, but it is by proper culture alone that they are cherished and brought to maturity. A few years of early and assiduous application

never fail to procure us the reward of our industry; and who that knows the pleasures and advantages which the sciences afford, would think his time, in this case, misspent or his labours useless? Riches and honours are the gifts of fortune, casually bestowed, or hereditarily received, and are frequently abused by their possessors; but the superiority of wisdom and knowledge is a pre-eminence of merit, which originates with the man, and is the noblest of all distinctions.

Nature, bountiful and wise in all things, has provided us with an infinite variety of scenes, both for our instruction and entertainment; and, like a kind and indulgent parent, admits all her children to an equal participation of her blessings. But, as the modes, situations, and circumstances of life are various, so accident, habit, and education, have each their predominating influence, and give to every mind its particular bias. Where examples of excellence are wanting, the attempts to attain it are but few; but eminence excites attention, and produces imitation. To raise the curiosity, and to awaken the listless and dormant powers of younger minds, we have only to point out to them a valuable acquisition, and the means of obtaining it; the active principles are immediately put into motion, and the certainty of the conquest is ensured from a determination to conquer.

But of all the sciences which serve to call forth this spirit of enterprise and inquiry, there are none more eminently useful than the Mathematics. By an early attachment to these elegant and sublime studies, we acquire a habit of reasoning, and an elevation of thought, which fixes the mind, and prepares it for every other pursuit. From a few simple axioms, and evident principles, we proceed gradually to the most general propositions, and remote analogies: deducing one truth from another, in a chain of argument well connected and logically pursued: which brings us at last, in the most satisfactory manner, to the conclusion, and serves as a general direction in all our inquiries after truth.

And it is not only in this respect that mathematical learning is so highly valuable; it is, likewise, equally estimable for its practical utility. Almost all the works of art and devices of man have a dependence upon its principles, and are indebted to it for their origin and perfection. The cultivation of these admirable sciences is, therefore, a

thing of the utmost importance, and ought to be considered as a principal part of every liberal and well-regulated plan of education. They are the guide of our youth, the perfection of our reason, and the foundation of every great and noble undertaking.

From these considerations, I have been induced to compose an introductory course of mathematical science; and, from the kind encouragement which I have hitherto received, am not without hopes of a continuance of the same candour and approbation. Considerable practice as a teacher, and a long attention to the difficulties and obstructions which retard the progress of learners in general, have enabled me to accommodate myself the more easily to their capacities and understandings. And as an earnest desire of promoting and diffusing useful knowledge is the chief motive for this undertaking, so no pains or attention shall be wanting to make it as complete and perfect as possible.

The subject of the present performance is ALGEBRA; which is one of the most important and useful branches of those sciences, and may be justly considered as the key to all the rest. Geometry delights us by the simplicity of its principles, and the elegance of its demonstrations: Arithmetic is confined in its object, and partial in its application; but Algebra, or the analytic art, is general and comprehensive, and may be applied with success in all cases where truth is to be obtained and proper data can be established.

To trace this science to its birth, and to point out the various alterations and improvements it has undergone in its progress, would far exceed the limits of a preface.* It will be sufficient to observe, that the invention is of great antiquity, and has challenged the praise and admiration of all ages. DIOPHANTUS, a Greek mathematician of Alexandria in Egypt, who flourished in or about the fourth century after CHRIST, appears to have been the first, among

* Those who are desirous of a knowledge of this kind, may consult the Introduction to my *Treatise on Algebra*, 2d Edition, 2 vols. 8vo. 1820, where they will find a regular historical detail of the rise and progress of the science, from its first rude beginnings to the present time; together with a variety of other particulars, relating to the theoretical and practical part of the subject, which are there more fully explained and developed, than could have been done in a compendium like the present.

the ancients, who applied it to the solution of indeterminate, or unlimited problems; but it is to the moderns that we are principally indebted for the most curious refinements of the art, and its great and extensive usefulness in every abstruse and difficult inquiry. NEWTON, MACLAURIN, SAUNDERSON, SIMPSON, and EMERSON, among our own countrymen, and CLAIRAUT, EULER, LAGRANGE, and LACROIX, on the continent, are those who have particularly excelled in this respect; and it is to their works that I would refer the young student, as the patterns of elegance and perfection.

The following compendium is formed entirely upon the model of those writers, and is intended as a useful and necessary introduction to them. Almost every subject, which belongs to pure Algebra, is concisely and distinctly treated of; and no pains have been spared to make the whole as easy and intelligible as possible. A great number of elementary books have already been written upon this subject; but there are none, which I have yet seen, but what appear to me to be extremely defective. Besides being totally unfit for the purpose of teaching, they are generally calculated to vitiate the taste, and mislead the judgment. A tedious and inelegant method prevails through the whole, so that the beauty of the science is generally destroyed by the clumsy and awkward manner in which it is treated; and the learner, when he is afterwards introduced to some of our best writers, is obliged, in a great measure, to unlearn and forget every thing which he has been at so much pains in acquiring.

There is a certain taste and elegance in the sciences, as well as in every branch of polite literature, which is only to be obtained from the best authors, and a judicious use of their instructions. To direct the student in his choice of books, and to prepare him properly for the advantages he may receive from them, is therefore the business of every writer who engages in the humble, but useful task of a preliminary tutor. This information I have been careful to give, in every part of the present performance, where it appeared to be in the least necessary; and though the nature and confined limits of my plan admitted not of diffuse observations, or a formal enumeration of particulars, it is presumed nothing of real use and importance has

been omitted. My principal object was to consult the ease, satisfaction, and accommodation of the learner; and the favourable reception the work has met with from the public, has afforded me the gratification of believing that my labours have not been unsuccessfully employed.

JOHN BONNYCASTLE.

ROYAL MILITARY ACADEMY,
WOOLWICH,
October 22, 1815.

ADVERTISEMENT TO THE THIRTEENTH EDITION.

CONSIDERABLE improvements having lately been made in the Solution of Equations by Approximation, a subject of great importance in Algebra, I have been induced to add an Addenda to the present Edition of this work, containing an entirely new method for that purpose; which I trust will be found, in many respects, more convenient than any hitherto published.

CHARLES BONNYCASTLE.

CHATHAM,
July 19, 1824.

EDITOR'S PREFACE.

In presenting a New Edition of Bonnycastle's Algebra to the public, the Editor indulges a hope that he has succeeded in the attempts which he has made to render the work deserving a continuance of that extensive support and encouragement which it has so long received, from the principal Educational Establishments in this Kingdom. From a careful scrutiny of all the algebraical processes throughout, several blemishes have been detected and removed, and, in a few instances, useful additions and emendations have been introduced; of the former may be mentioned the Synopsis on Variable Quantities, forming an Appendix to the book, which it is hoped will prove acceptable to the young Matnematician. Upon the whole the Editor trusts that this *Sixteenth Edition* will be found on examination to retain its character as an useful work, so as to merit a continuance of public favour.

SAMUEL MAYNARD.

No. 8, Earl's Court, Leicester Square,
London; December, 16*th,* 1835.

ALGEBRA.

ALGEBRA is the science which treats of a general method of performing calculations, and resolving mathematical problems, by means of the letters of the alphabet.

Its leading rules are the same as those of arithmetic; and the operations to be performed are denoted by the following characters :

$+$ *plus*, or *more*, the sign of addition; signifying that the quantities between which it is placed are to be added together.

Thus, $a+b$ shows that the number, or quantity, represented by b, is to be added to that represented by a, and is read a plus b.

$-$ *minus*, or *less*, the sign of subtraction ; signifying that the latter of the two quantities between which it is placed is to be taken from the former.

Thus, $a-b$ shows that the number, or quantity, represented by b is to be taken from that represented by a; and is read a minus b.

Also, $a \sim b$ represents the difference of the two quantities a and b, when it is not known which of them is the greater.

$\times$ *into*, the sign of multiplication; signifying that the quantities between which it is placed are to be multiplied together.

Thus, $a \times b$ shows that the number, or quantity, represented by a is to be multiplied by that represented by b; and is read a into b.

The multiplication of simple quantities is also frequently denoted by a point, or by joining the letters together in the form of a word.

Thus, $a \times b$, $a.b$, and ab, all signify the product of a and b: also, $3 \times a$, or $3a$, is the product of 3 and a; and is read 3 times a.

$\div$ *by*, the sign of division ; signifying that the former of the two quantities between which it is placed is to be divided by the latter.

Thus, $a \div b$ shows that the number, or quantity, represented by a is to be divided by that represented by b; and is read a by b, or a divided by b.

Division is also frequently denoted by placing one of the two quantities over the other, in the form of a fraction.

Thus, $b \div a$ and $\frac{b}{a}$ both signify the quotient of b divided by a; and $\frac{a-b}{a+c}$ signifies that $a-b$ is to be divided by $a+c$.

$=$ *equal to*, the sign of equality; signifying that the quantities between which it is placed are equal to each other.

Thus, $x = a + b$ shows that the quantity denoted by x is equal to the sum of the numbers, or quantities, a and b; and is read x equal to a plus b.

$\equiv$ *identical to*, or the sign of equivalence; signifying that the expressions between which it is placed are equal for all values of the letters of which they are composed.

Thus, $\frac{1}{2}(a+x)+\frac{1}{2}(a-x) \equiv a$; $\frac{1}{2}(a+x)-\frac{1}{2}(a-x) \equiv x$; and $(x+a) \times (x-a) \equiv x^2-a^2$, whatever numeral values may be given to the quantities represented by x and a.

$>$ *greater than*, the sign of majority; signifying that the former of the two quantities between which it is placed is greater than the latter.

Thus, $a > b$ shows that the number, or quantity, represented by a is greater than that represented by b; and is read a greater than b.

$<$ *less than*, the sign of minority; signifying that the former of the two quantities between which it is placed is less than the latter.

Thus, $a < b$ shows that the number, or quantity, represented by a is less than that represented by b; and is read a less than b.

$:$ *as*, or *to*, and $::$ *so is*, the signs of an equality of ratios; signifying that the quantities between which they are placed are proportional.

Thus, $a:b::c:d$ denotes that a has the same ratio to b that c has to d, or that a, b, c, d, are proportionals; and is read as a is to b so is c to d, or a is to b as c is to d.

$\surd$ *the radical sign*, signifying that the quantity before which it is placed is to have some root of it extracted.

Thus, $\sqrt{a}$ is the square root of a; $\sqrt[3]{a}$ is the cube root of a; $\sqrt[4]{a}$ is the fourth root of a; and so on.

The roots of quantities are also frequently represented by figures placed at the right-hand corner of them, in the form of a fraction.

Thus, $a^{\frac{1}{2}}$ is the square root of a; $a^{\frac{1}{3}}$ is the cube root of a; $a^{\frac{1}{4}}$ is the fourth root of a; and $a^{\frac{1}{n}}$, or $\sqrt[n]{a}$, is the nth root of a, or a root denoted by any number n.

In like manner, a^2 is the square of a; a^3 is the cube of a; a^4 is the fourth power of a; and a^m is the mth power of a, or any power denoted by the number m.

∞ is the sign of infinity; signifying that the quantity standing before it is of an unlimited value, or greater than any quantity that can be assigned.

The coefficient of a quantity is the number or letter prefixed to it; being that which shows how often the other is to be taken.

Thus, in the quantities $3b$ and $-\frac{2}{3}b$, 3 and $-\frac{2}{3}$ are the coefficients of b; and a is the coefficient of x in the quantity ax.

A quantity without any coefficient prefixed to it is supposed to have 1, or unity; and when a quantity has no sign before it, $+$ is always understood.

Thus, a is the same as $+a$, or $+1a$; and $-a$ is the same as $-1a$.

A term is any part or member of a compound quantity which is separated from the rest by the signs $+$ or $-$.

Thus, a and b are the terms of $a+b$; and $3a$, $-2b$, and $+5cd$, are the terms of $3a-2b+5cd$.

In like manner, the terms of a product, fraction, or proportion, are the several parts or quantities of which they are composed.

Thus, a and b are the terms of ab, or of $\frac{a}{b}$; and a, b, c, d, are the terms of the proportion $a:b::c:d$.

Factors are the numbers, or quantities, from the multiplication of which other numbers, or quantities, are produced.

Thus, a and b are the factors of ab; also, 2, a, and b^2, are the factors of $2ab^2$; and $a+x$ and $a-x$ are the factors of the product $(a+x)\times(a-x)$.

Like quantities, are those which consist of the same letters or combinations of letters, or that differ only in their coefficients; as a and $3a$, or $5ab$ and $7ab$, or $2a^2b$ and $9a^2b$.

Unlike quantities, are those which consist of different letters or combinations of letters; as a and b, or $3a$ and a^2, or $5ab^2$ and $7a^2b$.

Given quantities, are such as have known values, and are generally represented by some of the first letters of the alphabet; as a, b, c, d, &c.

Unknown quantities, are such as have no fixed or determinate values; and are usually represented by some of the final letters of the alphabet; as x, y, z.

Simple quantities, are those which consist of one term only; as $3a$, $5ab$, $-8a^2b$, &c.

Compound quantities, are those which consist of several terms; as $2a+b$, or $3a-2c$, or $a+2b-3c$, &c.

Positive, or affirmative quantities, are those that are to be added; or such as stand simply by themselves, or have the sign $+$ prefixed to them; as a, or $+a$, or $+3ab$, &c.

Negative quantities, are those that are to be subtracted; or such as have the sign $-$ prefixed to them; as $-a$, or $-3ab$, or $-7ab^2$, &c.

Like signs, are such as are all positive, or all negative; as $+$ and $+$, or $-$ and $-$.

Unlike signs, are when some are positive, and others negative; as $+$ and $-$, or $-$ and $+$.

A monomial, is a quantity consisting of one term; as a, $2b$, $-3a^2b$, &c., being the same as a simple quantity, or one that stands by itself, without any connexion with others.

A binomial, is a quantity consisting of two terms; as $a+b$, or $a-b$; the latter of which is, also, sometimes called a residual quantity.

A trinomial, is a quantity consisting of three terms, as $a+2b-3c$; a quadrinomial of four, as $a-2b+3c-d$; and a polynomial, or multinomial, is that which has many terms.

The power of a quantity, is its square, cube, biquadrate, &c.; called also its second, third, fourth power, &c.; as a^2, a^3, a^4, &c.

The index, or exponent of a quantity, is the number which denotes its power or root.

Thus, -1 is the index, or exponent, of a^{-1}, 2 is the index of a^2; $\frac{1}{2}$ of $a^{\frac{1}{2}}$ or $\sqrt{a}$; and m and $\frac{1}{n}$ of a^m and $a^{\frac{1}{n}}$.

When a quantity appears without any index, or exponent, it is always understood to have unity, or 1.

Thus, a is the same as a^1, and $2x$ is the same as $2x^1$; the 1, in such cases, being usually omitted.

A rational number, or quantity, is that which can be expressed in finite terms, or without any radical sign, or fractional index; as a, or $\frac{2}{3}a$, or $5a^2$, &c.

An irrational quantity, or surd, is that which has no exact root, or which can only be expressed by means of the radical sign, or a fractional index; as $\sqrt{2}$ or $2^{\frac{1}{2}}$, $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$, &c.

A square or cube number, &c., is that which has an exact square or cube root, &c.

Thus, 4 and $\frac{9}{16}a^2$ are square numbers; and 64 and $\frac{8}{27}a^3$ are cube numbers, &c.

A measure, or divisor, of any quantity, is that which is contained in it some exact number of times.

Thus, 3 is a measure, or divisor, of 6, $7a$ is a measure of $35a$, and $9ab$ of $27a^2b^2$.

A composite number, or quantity, is that which is produced by the multiplication of two or more terms or factors.

Thus, 6 is a composite number, formed of the factors 2 and 3, or 2×3; and $3abc$ is a composite quantity, the factors of which are 3, a, b, c.

Commensurable numbers, or quantities, are such as can be each divided by the same quantity, without leaving a remainder.

Thus, 6 and 8, $2\sqrt{2}$, and $3\sqrt{2}$, $5a^2b$ and $7ab^2$, are commensurable quantities; the common divisors being 2, $\sqrt{2}$, and ab.

Incommensurable numbers, or quantities, are such as have no common measure, or divisor, except unity.

Thus, 2 and 7, 5 and 8, $\sqrt{2}$ and $\sqrt{3}$, and $a+b$ and a^2+b^2, are incommensurable quantities.

Also, when two numbers have no common measure, or divisor, except unity, they are said to be prime to each other; as is the case with the numbers 2 and 7, or 5 and 8.

A multiple of any quantity, is that which is some exact number of times that quantity.

Thus, 12 is a multiple of 4, $15a$ is a multiple of $3a$, and $20a^2b^2$ of $5ab$.

The reciprocal of any quantity, is that quantity inverted, or unity divided by it.

Thus, the reciprocal of a, or $\frac{a}{1}$, is $\frac{1}{a}$; and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

A function of one or more quantities, is an expression into which those quantities enter, in any manner whatever, either combined, or not, with known quantities.

Thus, $a-2x$, $ax+3x^2$, $2x-a(a^2-x^2)^{\frac{1}{2}}$, ax^m, a^x, &c. are functions of x; and $axy\times bx^2$, $ay+x(ax^2-by^3)^{\frac{1}{2}}$, &c. are functions of x and y.

A vinculum, is a bar ——, or parenthesis ( ), made use of to collect several quantities into one.

Thus, $\overline{a+b}\times c$, or $(a+b)c$, denotes that the compound quantity $a+b$ is to be multiplied by the simple quantity c; and $\sqrt{(ab+c^2)}$, or $(ab+c^2)^{\frac{1}{2}}$ is the square root of the compound quantity $ab+c^2$.

PRACTICAL EXAMPLES

For computing the numeral Values of Algebraic Expressions.

Supposing $a=6$, $b=5$, $c=4$, $d=1$, and $e=0$.

Then

1. $a^2+2ab-c+d=36+60-4+1=93$.
2. $2a^3-3a^2b+c^3=432-540+64=-44$.
3. $a^2\times(a+b)-2abc=36\times(6+5)-240=(36\times11)-240=156$.
4. $2a\sqrt{(b^2-ac)}+\sqrt{(2ac+c^2)}=12\sqrt{(25-24)}+\sqrt{(48+16)}=12\sqrt{1}+\sqrt{64}=12+8=20$.

5. $3a\sqrt{(2ac+c^2)}$, or $3a(2ac+c^2)^{\frac{1}{2}}=18\sqrt{(48+16)}=18\sqrt{64}=18\times 8=144$.

6. $\dfrac{2a+3c}{6d+4e}+\dfrac{4bc}{\sqrt{2ac+c^2}}=\dfrac{12+12}{6+0}+\dfrac{80}{\sqrt{(48+16)}}=\dfrac{24}{6}+\dfrac{80}{8}=14$.

7. $\sqrt{(2a^2-\sqrt{2ac+c^2})}=\sqrt{(72-\sqrt{64})}=\sqrt{(72-8)}=\sqrt{64}=8$.

EXAMPLES FOR PRACTICE.

Required the numeral values of the following expressions; supposing a, b, c, d, e, to be 6, 5, 4, 1, and 0, respectively, as before.

1. $2a^2+3bc-5d$
2. $5a^2b-10ab^2+2e$
3. $7a^2+b-c\times d+e$
4. $5\sqrt{ab}+b^2-2ab-e^2$
5. $\dfrac{a}{c}\times d-\dfrac{a-b}{d}+2a^2e$
6. $3\sqrt{c}+2a\sqrt{(2a+b-d)}$
7. $a\sqrt{a^2+b^2}+3bc\sqrt{(a^2-b^2)}$
8. $3a^2b+\sqrt[3]{(c^2+\sqrt{2ac+c^2})}$
9. $\dfrac{2b+c}{3a-c}-\dfrac{\sqrt{5b}+3\sqrt{c+d}}{2a+c}$

ADDITION

Addition is the connecting of quantities together by means of their proper signs, and incorporating such as are like, or that can be united, into one sum; the rule for performing which is commonly divided into the three following cases:*

CASE I.

When the Quantities are like, and have like Signs.

RULE.

Add all the coefficients of the several quantities together, and to their sum annex the letter or letters belonging to each term, prefixing, when necessary, the common sign.

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* The term Addition, which is generally used to denote this rule, is too scanty to express the nature of the operations that are to be performed in it; which are sometimes those of addition, and sometimes subtraction, according as the quantities are negative or positive. It should, therefore, be called by some name signifying incorporation, or striking a balance; in which case, the incongruity, here mentioned, would be removed.

EXAMPLES.

(1)	(2)	(3)
$3a$	$-3ax$	$2b+3y$
$5a$	$-6ax$	$5b+7y$
a	$-ax$	$b+2y$
$7a$	$-2ax$	$8b+y$
$12a$	$-7ax$	$4b+4y$
$28a$	$-19ax$	$20b+17y$

(4)	(5)	(6)
$2ay$	$-2by^2$	$a-2x^2$
$5ay$	$-6by^2$	$a-6x^2$
$4ay$	$-by^2$	$4a-x^2$
$7ay$	$-8by^2$	$3a-5x^2$
$16ay$	$-by^2$	$7a-x^2$

(7)	(8)	(9)
$3ax^2$	$7x-4y$	$2a+x^2$
$2ax^2$	$x-8y$	$3a+x^2$
$12ax^2$	$3x-y$	$a+2x$
$9ax^2$	$x-3y$	$9a+3x^2$
$10ax^2$	$4x-y$	$4a+x^2$

CASE II.

When the Quantities are like, but have unlike Signs.

RULE.

Add all the affirmative coefficients into one sum, and those that are negative into another, when there are several of the same kind; then subtract the less of these results from the greater, and prefix the sign of the greater to the difference, annexing the common letter or letters, as before.

EXAMPLES.

(1)	(2)	(3)
$-\ 3a$	$2a-3x^2$	$6x+5ay$
$+\ 7a$	$-7a+5x^2$	$-3x+2ay$
$+\ 8a$	$-3a+x^2$	$x-6ay$
$-\ a$	$+a-3x^2$	$2x+ay$
$+11a$	$-7a\quad *$	$6x+2ay$

(4)	(5)	(6)
$-\ 2a^2$	$2ay-7$	$-3ab+7x$
$-\ 3a^2$	$-ay+8$	$+3ab-10x$
$-\ 8a^2$	$+2ay-9$	$+3ab-6x$
$+10a^2$	$-3ay-11$	$-ab+2x$
$+16a^2$	$+12ay+13$	$+2ab+7x$

(7)	(8)	(9)
$-2a\sqrt{x}$	$-6a^2+2b$	$6ax^2+5x^{\frac{1}{2}}$
$+a\sqrt{x}$	$+2a^2-3b$	$-2ax^2-6x^{\frac{1}{2}}$
$-3a\sqrt{x}$	$-5a^2-8b$	$+3ax^2-10x^{\frac{1}{2}}$
$+7a\sqrt{x}$	$+4a^2-2b$	$-7ax^2+3x^{\frac{1}{2}}$
$-4a\sqrt{x}$	$-3a^2+9b$	$+ax^2+11x^{\frac{1}{2}}$

CASE III.

When the Quantities are unlike; or some like and others unlike.

RULE.

Collect all the like quantities together, by taking their sums or differences, as in the foregoing cases, and set down those that are unlike, one after another, with their proper signs.

EXAMPLES.

(1)	(2)	(3)
$\begin{array}{r} 5xy \\ 4ax \\ -xy \\ -4ax \\ \hline 4xy \\ \hline \end{array}$	$\begin{array}{r} 2xy-2x^2 \\ 3x^2+xy \\ x^2+xy \\ 4x^2-3xy \\ \hline 6x^2+xy \\ \hline \end{array}$	$\begin{array}{l} 2ax-30 \\ 3x^2-2ax \\ 5x^2-3x^{\frac{1}{2}} \\ 3\sqrt{x}+10 \\ \hline 8x^2-20 \\ \hline \end{array}$

(4)	(5)	(6)
$\begin{array}{l} +ax^{\frac{1}{2}} \\ -ax^2 \\ +3ax^2 \\ -ax^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$	$\begin{array}{l} 8a^2x^2-3ax \\ 7ax-5xy \\ 9xy-5ax \\ 2a^2x^2+xy \\ \hline \\ \hline \end{array}$	$\begin{array}{l} 10b^2-3a^2x \\ -b^2+2a^2x^2 \\ 50+2a^2x \\ a^2x^2+120 \\ \hline \\ \hline \end{array}$

(7)	(8)	(9)
$\begin{array}{l} +3x^2y \\ -2xy^2 \\ -3xy^2 \\ -8x^2y \\ +2xy^2 \\ \hline \\ \hline \end{array}$	$\begin{array}{l} 2\sqrt{x}-18y \\ 3\sqrt{xy}+10x \\ 2x^2y+25y \\ 12xy-\sqrt{xy} \\ -8y+17x^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$	$\begin{array}{l} 2a^2-3a\sqrt{x} \\ x^2-2a^{\frac{1}{2}}x^{\frac{1}{2}} \\ 3a^2-13xy \\ xy+32a^2 \\ 20-65x^2 \\ \hline \\ \hline \end{array}$

EXAMPLES FOR PRACTICE.

1. It is required to find the sum of $\frac{a+b}{2}$ and $\frac{a-b}{2}$.
2. Add $5x-3a+b+7$ and $-4a-3x+2b-9$ together.
3. Add $2a+3b-4c-9$ and $5a-3b+2c-10$ together.
4. Add $3a+2b-5$, $a+5b-c$, and $6a-2c+3$ together.
5. Add $2x+ax$, and $3x^2-bx$ together.
6. Add x^3+ax^2+bx+2 and x^3+cx^2+dx-1 together.

SUBTRACTION.

SUBTRACTION is the taking of one quantity from another; or the method of finding the difference between any two quantities of the same kind; which is performed as follows :*

RULE.

Change all the signs (+ and −) of the lower line, or quantities that are to be subtracted, into the contrary signs, or rather conceive them to be so changed, and then collect the terms together, as in the several cases of addition.

EXAMPLES.

$$\begin{array}{ccc}
(1) & (2) & (3) \\
\begin{array}{r} 5a^2-2b \\ 2a^2+5b \\ \hline 3a^2-7b \\ \hline \end{array} &
\begin{array}{r} x^2-\ 2y+3 \\ 4x^2+\ 9y-5 \\ \hline -3x^2-11y+8 \\ \hline \end{array} &
\begin{array}{r} 5xy+\ 8x-2 \\ 3xy-\ 8x-7 \\ \hline 2xy+16x+5 \\ \hline \end{array}
\end{array}$$

$$\begin{array}{ccc}
(4) & (5) & (6) \\
\begin{array}{r} 5xy-18 \\ -xy+12 \\ \hline \\ \hline \end{array} &
\begin{array}{r} 8y^2-2y-5 \\ -y^2+3y+2 \\ \hline \\ \hline \end{array} &
\begin{array}{r} 10-8x-3xy \\ -x+3\ -\ xy \\ \hline \\ \hline \end{array}
\end{array}$$

$$\begin{array}{ccc}
(7) & (8) & (9) \\
\begin{array}{r} -5x^2y-8a \\ +3x^2y-7b \\ \hline \\ \hline \end{array} &
\begin{array}{r} 4\sqrt{ax}-2x^2y \\ 3\sqrt{ax}-5xy^2 \\ \hline \\ \hline \end{array} &
\begin{array}{r} 5x^2+\sqrt{x}-4y \\ 6x^2-\ 8x-x^{\frac{1}{2}} \\ \hline \\ \hline \end{array}
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the difference of $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b.)$
2. From $3x-2a-b+7$, subtract $8-3b+a+4x$.
3. From $3a+b+c-2d$, subtract $b-8c+2d-8$.

---

* The term subtraction, used for this rule, is liable to the same objection as that for addition; the operations to be performed being frequently of a mixed nature, like those of the former.

4. From $13x^2-2ax+9b^2$, subtract $5x^2-7ax-b^2$.
5. From $20ax-5\sqrt{x}+3a$, subtract $4ax+5x^{\frac{1}{2}}-a$.
6. From $5ab+2b^2-c+bc-b$, take $b^2-2ab+bc$
7. From $3x^2+ax+2$, subtract $2x^2+bx-4$.
8. From ax^3-bx^2+cx-d, subtract $bx^2+ex-2d$.

MULTIPLICATION.

Multiplication, or the method of finding the product of two or more quantities, is performed in the same manner as in arithmetic; except that it is usual, in this case, to begin the operation at the left hand, and to proceed towards the right, or contrary to the way employed in the latter.

The rule is commonly divided into three cases; in each of which it is necessary to observe, that like signs, in multiplying, produce $+$, and unlike signs, $-$.

It is likewise to be remarked, that powers, or roots of the same quantity, are multiplied together by adding their indices: Thus,

$a\times a^2$, or $a^1\times a^2=a^3$; $a^2\times a^3=a^5$; $a^{\frac{1}{2}}$* $\times a^{\frac{1}{3}}=a^{\frac{5}{6}}$; and $a^m\times a^n=a^{m+n}$; where m and n may be either integers or fractions.

The multiplication of compound quantities is also, sometimes, barely denoted by writing them down, with their proper signs, under a vinculum, or a parenthesis, without performing the whole operation, as

$$3ab(a-b), \text{ or } 2a\sqrt{(a^2+b^2)} \text{ or } a\sqrt[3]{(a+b)}.$$

This method is often preferable to that of executing the entire process; particularly when the product of two or more factors is to be divided by some other quantity, because, in this case, any quantity that is common to both the divisor and dividend may be more readily suppressed; as will be evident from various instances in the following part of the work.†

---

* According to addition of fractions $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$.

† The above rule for the signs may be proved thus: If B, b, be any two quantities, of which B is the greater, and B$-b$ is to be multiplied by a, it is plain that the product, in this case, must be less than a B,

CASE I.

When the Factors are both simple Quantities.

RULE.

Multiply the coefficients of the two terms together, and to the product annex all the letters, or their powers, belonging to each, after the manner of a word; and the result, with the proper sign prefixed, will be the product required.*

EXAMPLES.

(1)	(2)	(3)	(4)
$12a$	$-2a$	$+5a$	$-9x^2$
$3b$	$+4b$	$-6x$	$-5bx$
$36ab$	$-8ab$	$-30ax$	$+45bx^3$

---

because $B-b$ is less than B; and, consequently, when each of the terms of the former are multiplied by a, as above, the result will be

$$(B-b)\times a=aB-ab.$$

For if it were $aB+ab$, the product would be greater than aB, which is absurd.

Also, if B be greater than b, and A greater than a, and it is required to multiply $B-b$ by $A-a$, the result will be

$$(B-b)\times(A-a)=AB-aB-bA+ab.$$

For the product of $B-b$ by A is $A\,(B-b)$, or $AB-Ab$, and that of $B-b$ by $-a$ is $-a\,(B-b)$, as has been before shown; whence $B-b$ being less than B, it is evident that $-a\,(B-b)$, or the part which is to be taken from $A\,(B-b)$, must be less than aB; and, consequently, since the first part of this product is $-aB$, the second part must be $+ab$; for if it were $-ab$, a greater part than aB would be to be taken from $A\,(B-b)$, which is absurd.

* When any number of quantities are to be multiplied together, it is the same thing in whatever order they are placed: thus, if ab is to be multiplied by c, the product is either abc, acb, or bca, &c.; although it is usual, in this case, as well as in addition and subtraction, to put them according to their rank in the alphabet. It may here also be observed, in conformity to the rule given above for the signs, that $(+a)\times(+b)$, or $(-a)\times(-b)=+ab$; and $(+a)\times(-b)$, or $(-a)\times(+b)=-ab$.

(5)	(6)	(7)	(8)
$7ab$	$-6a^2x$	$-2xy^2$	$-7axy$
$-5ac$	$+5x$	$-xy$	$+6ay$

(9)	(10)	(11)	(12)
$3a^2b$	$12a^2x$	$-6xyz$	$-a^2xy$
$2a^2b$	$-2x^2y$	$+ay^2z$	$+2xy^2$

CASE II.

When one of the Factors is a compound Quantity.

RULE.

Multiply every term of the compound factor, considered as a multiplicand, separately, by the multiplier, as in the former case; then these products placed one after another, with their proper signs, will be the whole product required.

EXAMPLES.

(1)	(2)	(3)
$3a-2b$	$6xy-8$	a^2-2x+1
$4a$	$3x$	$4x$
$12a^2-8ab$	$18x^2y-24x$	$4a^2x-8x^2+4x$

(4)	(5)	(6)
$12x-ab$	$35x^2-7a$	$3y^2+y-2$
$5a$	$-2x$	xy

(7)	(8)	(9)
$13x^2-a^2b$	$25xy+3a^2$	$3x^2-xy-2y^2$
$-2a$	$13x^2$	$5x^2y$

CASE III.

When both the Factors are compound Quantities.

RULE.

Multiply every term of the multiplicand, separately, by each term of the multiplier, setting down the products one after another, with their proper signs; then add the several lines of products together, and their sum will be the whole product required.

EXAMPLES

(1)

$$\begin{array}{l} a+x \\ a+x \\ \hline a^2+ax \\ \quad +ax+x^2 \\ \hline a^2+2ax+x^2 \\ \hline \end{array}$$

(2)

$$\begin{array}{l} 5x+\ 4y \\ 3x-\ 2y \\ \hline 15x^2+12xy \\ \qquad -10xy-8y^2 \\ \hline 15x^2+\ 2xy-8y^2 \\ \hline \end{array}$$

(3)

$$\begin{array}{l} x^2+xy-y^2 \\ x-y \\ \hline x^3+x^2y-xy^2 \\ \quad -x^2y-xy^2+y^3 \\ \hline x^3\ \ *\ \ -2xy^2+y^3 \\ \hline \end{array}$$

(4)

$$\begin{array}{l} a+x \\ a-x \\ \hline a^2+ax \\ \quad -ax-x^2 \\ \hline a^2\ \ *\ \ -x^2 \\ \hline \end{array}$$

(5)

$$\begin{array}{l} x^2+y \\ x^2+y \\ \hline x^4+\ x^2y \\ \quad +\ x^2y+y^2 \\ \hline x^4+2x^2y+y^2 \\ \hline \end{array}$$

(6)

$$\begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3\ \ *\ \ *\ \ -y^3 \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the product of $x^2-xy^2+y^2$ and $x+y$.
   Ans. x^3+y^3.
2. Required the product of $x^3+x^2y+xy^2+y^3$ and $x-y$.
   Ans. x^4-y^4.
3. Required the product of x^2+xy+y^2 and x^2-xy+y^2.
   Ans. $x^4+x^2y^2+y^4$.
4. Required the product of $3x^2-2xy+5$ and $x^2+2xy-3$.
   Ans. $3x^4+4x^3y-4x^2y^2-4x^2+16xy-15$.

5. Required the product of $2a^2-3ax+4x^2$ and $5a^2-6ax-2x^2$. *Ans.* $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$.

6. Required the product of $5x^3+4ax^2+3a^2x+a^3$ and $2x^2-3ax+a^2$. *Ans.* $10x^5-7ax^4-a^2x^3-3a^3x^2+a^5$.

7. Required the product of $3x^3+2x^2y^2+3y^3$ and $2x^3-3x^2y^2+5y^3$. *Ans.* $6x^6-5x^5y^2-6x^4y^4+21x^3y^3+x^2y^5+15y^6$.

8. Required the product of x^3-ax^2+bx-c and x^2-dx+e. *Ans.* $x^5-(a+d)x^4+(b+ad+e)x^3-(c+bd+ae)x^2$
$+(cd+be)x-ce$.

DIVISION

Division is the converse of multiplication, and is performed like that of numbers; the rule being usually divided into three cases; in each of which, like signs give + in the quotient, and unlike signs −, as in finding their products.*

It is here also to be observed, that powers and roots of the same quantity are divided by subtracting the index of the divisor from that of the dividend.

Thus, $a^3 \div a^2$, or $\dfrac{a^3}{a^2}=a$; $a^{\frac{1}{2}} \div a^{\frac{1}{3}}$, or $\dfrac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}}=a^{\frac{1}{6}}$; $a^{\frac{3}{4}} \div a^{\frac{2}{3}}$ or $\dfrac{a^{\frac{3}{4}}}{a^{\frac{2}{3}}}=a^{\frac{1}{12}}$; and $a^m \div a^n$, or $\dfrac{a^m}{a^n}=a^{m-n}$.

CASE I.

When the Divisor and Dividend are both simple Quantities.

RULE.

Set the dividend over the divisor, in the manner of a

---

* According to the rule here given for the signs, it follows that

$$\frac{+ab}{+b}=+a;\ \frac{-ab}{-b}=+a;\ \frac{-ab}{+b}=-a;\frac{+ab}{-b}=-a;$$

as will readily appear by multiplying the quotient by the divisor; the signs of the products being then the same as would take place in the former rule.

fraction, and reduce it to its simplest form, by cancelling the letters and figures that are common to each term.

EXAMPLES.

$6ab \div 2a$, or $\dfrac{6ab}{2a} = 3b$; and $12ax^2 \div 3x$, or $\dfrac{12ax^2}{3x} =$ $4ax$; $a \div a$, or $\dfrac{a}{a} = 1$; and $a \div - a$, or $\dfrac{a}{-a} = -1$.

Also $-2a \div 3a$, or $\dfrac{-2a}{3a} = -\dfrac{2}{3}$; and $9x^{\frac{1}{2}} \div 3x^{\frac{1}{4}}$, or $\dfrac{9x^{\frac{1}{2}}}{3x^{\frac{1}{4}}} = 3x^{\frac{1}{2} - \frac{1}{4}} = 3x^{\frac{2}{4} - \frac{1}{4}} = 3x^{\frac{1}{4}}$.

1. Divide $16x^2$ by $8x$, and $12a^2x^2$ by $-8a^2x$.

   Ans. $2x$ and $-\dfrac{3x}{2}$.

2. Divide $-15ay^2$ by $3ay$, and $-18ax^2y$ by $-8ax$.

   Ans. $-5y$, and $2\frac{1}{4}xy$.

3. Divide $-\frac{2}{3}a^{\frac{1}{2}}$ by $\frac{4}{5}a^{\frac{1}{2}}$, and $ax^{\frac{1}{3}}$ by $-\frac{3}{5}a^{\frac{1}{2}}x^{\frac{1}{4}}$.

   Ans. $-\dfrac{5}{6}$, and $-\dfrac{5}{3}a^{\frac{1}{2}}x^{\frac{1}{12}}$

CASE II.

When the Divisor is a simple Quantity, and the Dividend a compound one.

RULE.

Divide each term of the dividend by the divisor, as in the former case; setting down such as will not divide in the simplest form they will admit of.

EXAMPLES.

$$(ab+b^2) \div 2b, \text{ or } \frac{ab+b^2}{2b} = \tfrac{1}{2}a + \tfrac{1}{2}b = \frac{a+b}{2}$$

$$(10ab - 15ax) \div 5a, \text{ or } \frac{10ab - 15ax}{5a} = 2b - 3x.$$

$$(30ax - 48x^2) \div 6x, \text{ or } \frac{30ax - 48x^2}{6x} = 5a - 8x.$$

1. Let $3x^3 + 6x^2 + 3ax - 15x$ be divided by $3x$.
Ans. $x^2 + 2x + a - 5$.

2. Let $3abc + 12abx - 9a^2b$ be divided by $3ab$.
Ans. $c + 4x - 3a$.

3. Let $40a^3b^3 + 60a^2b^2 - 17ab$ be divided by $-ab$.
Ans. $-40a^2b^2 - 60ab + 17$.

4. Let $15a^2bc - 12acx^2 + 5ad^2$ be divided by $-5ac$.
Ans. $-3ab + 2\frac{2}{5}x^2 - \dfrac{d^2}{c}$.

5 Let $20ax + 15ax^2 + 10ax + 5a$ be divided by $5a$.
Ans. $3x^2 + 6x + 1$.

CASE III.

When the Divisor and Dividend are both compound Quantities.

1. Set the quantities down in the same manner as in division of numbers, ranging the terms of each of them so that the higher powers of one of the letters may stand before the lower.

2. Divide the first term of the dividend by the first term of the divisor, and set the result in the quotient, with its proper sign, or simply by itself, if it be affirmative.

3. Multiply the whole divisor by the term thus found; and having subtracted the result from the dividend, bring down as many terms to the remainder as are requisite for the next operation, which perform as before; and so on, till the work is finished, as in common arithmetic.

EXAMPLES.

$$
\begin{array}{l}
x+y)x^2+2xy+y^2(x+y \\
\quad\ \ \underline{x^2+xy} \\
\qquad\quad xy+y^2 \\
\qquad\quad \underline{xy+y^2} \\
\qquad\qquad\ *
\end{array}
$$

$$
\begin{array}{l}
a+x)a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2 \\
\quad\ \ \underline{a^3+a^2x} \\
\qquad\quad 4a^2x+5ax^2 \\
\qquad\quad \underline{4a^2x+4ax^2} \\
\qquad\qquad\qquad ax^2+x^3 \\
\qquad\qquad\qquad \underline{ax^2+x^3} \\
\qquad\qquad\qquad\quad\ *
\end{array}
$$

$$
\begin{array}{l}
x-3)x^3-9x^2+27x-27(x^2-6x+9 \\
\qquad \underline{x^3-3x^2} \\
\quad\ -6x^2+27x \\
\quad\ \underline{-6x^2+18x} \\
\qquad\qquad 9x-27 \\
\qquad\qquad \underline{9x-27} \\
\qquad\qquad\quad\ *
\end{array}
$$

$$
\begin{array}{l}
2x^2-3ax+a^2)4x^4-9a^2x^2+6a^3x-a^4(2x^2+3ax-a^2 \\
\qquad\qquad\quad \underline{4x^4-6ax^3+2a^2x^2} \\
\qquad\qquad\qquad\quad 6ax^3-11a^2x^2+6a^3x \\
\qquad\qquad\qquad\quad \underline{6ax^3-\ \ 9a^2x^2+3a^3x} \\
\qquad\qquad\qquad\qquad\quad -\ \ 2a^2x^2+3a^3x-a^4 \\
\qquad\qquad\qquad\qquad\quad \underline{-\ \ 2a^2x^2+3a^3x-a^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad *
\end{array}
$$

NOTE 1. If the divisor be not exactly contained in the dividend, the quantity that remains after the division is finished must be placed over the divisor at the end of the quotient in the form of a fraction: Thus *

$$\begin{array}{l}
a+x)\,a^3-x^3\,(a^2-ax+x^2-\dfrac{2x^3}{a+x} \\
\qquad\ \ \underline{a^3+a^2x} \\
\qquad\quad -a^2x-x^3 \\
\qquad\quad \underline{-a^2x-ax^2} \\
\qquad\qquad\qquad ax^2-x^3 \\
\qquad\qquad\qquad \underline{ax^2+x^3} \\
\qquad\qquad\qquad\quad \underline{-2x^3}
\end{array}$$

$$\begin{array}{l}
x+y)\,x^4+y^4\,(x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y} \\
\qquad\ \ \underline{x^4+x^3y} \\
\qquad\quad -x^3y+y^4 \\
\qquad\quad \underline{-x^3y-x^2y^2} \\
\qquad\qquad\qquad x^2y^2+y^4 \\
\qquad\qquad\qquad \underline{x^2y^2+xy^3} \\
\qquad\qquad\qquad\quad -xy^3+y^4 \\
\qquad\qquad\qquad\quad \underline{-xy^3-y^4} \\
\qquad\qquad\qquad\qquad\quad \underline{2y^4}
\end{array}$$

2. The division of quantities may also be sometimes carried on, *ad infinitum*, like a decimal fraction; in which case a few of the leading terms of the quotient will generally be sufficient to indicate the rest, without its being necessary to continue the operation: thus,

---

* In the case here given, the operation of division may be considered as terminated, when the highest power of the letter, in the first, or leading term of the remainder, by which the process is regulated, is less than the power of the first term of the divisor; or when the first term of the divisor is not contained in the first term of the remainder; as the succeeding part of the quotient, after this, instead of being integral, as it ought to be, would necessarily become fractional; and, consequently, when this happens, the quotient must be completed in the manner above mentioned

$$
\begin{array}{l}
a+x)\,a\ldots\ldots\left(1-\dfrac{x}{a}+\dfrac{x^2}{a^2}-\dfrac{x^3}{a^3}+\dfrac{x^4}{a^4}-\text{\&c.}\right.\\
\quad\underline{a+x}\\
\quad\quad -x\\
\quad\quad \underline{-x-\dfrac{x^2}{a}}\\
\quad\quad\quad\quad +\dfrac{x^2}{a}\\
\quad\quad\quad\quad \underline{+\dfrac{x^2}{a}+\dfrac{x^3}{a^2}}\\
\quad\quad\quad\quad\quad\quad -\dfrac{x^3}{a^2}\\
\quad\quad\quad\quad\quad\quad \underline{-\dfrac{x^3}{a^2}-\dfrac{x^4}{a^3}}\\
\quad\quad\quad\quad\quad\quad\quad\quad \underline{+\dfrac{x^4}{a^3}}
\end{array}
$$

And by a process similar to the above it may be shown that

$$\frac{a}{a-x}=1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4}+\frac{x^5}{a^5}+\text{\&c.}$$

Whence the law, by which either of these series may be con tinued at pleasure, is obvious.

EXAMPLES FOR PRACTICE.

1. Let $a^2-2ax+x^2$ be divided by $a-x$. *Ans.* $a-x$
2. Let $x^3-3ax^2+3a^2x-a^3$ be divided by $x-a$.
   Ans. $x^2-2ax+a^2$.
3. Let $a^3+5a^2x+5ax^2+x^3$ be divided by $a+x$.
   Ans. $a^2+4ax+x^2$
4. Let $2y^3-19y^2+26y-17$ be divided by $y-8$.
   Ans. $2y^2-3y+2-\dfrac{1}{y-8}$.
5. Divide x^5+1 by $x+1$, and x^6-1 by $x-1$.
   Ans. $x^4-x^3+x^2-x+1$, and $x^5+x^4+x^3+x^2+x+1$.

6. Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.
Ans. $24x^2-2ax-35a^2$.

7. Let $4x^4-9x^2+6x-3$ be divided by $2x^2+3x-1$.
Ans. $2x^2-3x+1-\frac{2}{2x^2+3x-1}$.

8. Let $x^4-a^2x^2+2a^3x-a^4$ be divided by x^2-ax+a^2.
Ans. x^2+ax-a^2.

9. Let $6x^4-96$ be divided by $3x-6$, and a^5+x^5 by $a+x$.
Ans. $2x^3+4x^2+8x+16$, and $a^4-a^3x+a^2x^2-ax^3+x^4$.

10. Let $32x^5+243$ be divided by $2x+3$, and x^6-a^6 by $x-a$. *Ans.* $16x^4-24x^3+36x^2-54x+81$, and $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$.

11. Let b^4-3y^4 be divided by $b-y$, and $a^4+4a^2b+16b^4$ by $a+2b$. *Ans.* $b^3+b^2y+by^2+y^3-\frac{2y^4}{b-y}$, and $a^3-2a^2b+4ab+4ab^2-8b^2+8b^3$.

12. Let x^2+px+q be divided by $x+a$, and x^3-px^2+qx-r by $x-a$. *Ans.* $x+(p-a)-\frac{ap}{x}+\frac{a^2p}{x^2}+$ &c. and $x^2+(a-p)x+(a^2-ap+q)+$ &c.

OF ALGEBRAIC FRACTIONS.

Algebraic fractions have the same names and rules of operation as numeral fractions in common arithmetic; and the methods of reducing them, in either of these branches, to their most convenient forms, are as follows:

CASE I.

To find the greatest common measure of the terms of a Fraction.

RULE.

1. Arrange the two quantities according to the order of their powers, and divide that which is of the highest

dimensions by the other, having first expunged any factor that may be contained in all the terms of the divisor, without being common to those of the dividend.

2. Divide this divisor by the remainder, simplified, if necessary, as before; and so on, for each successive remainder, and its preceding divisor, till nothing remains; when the divisor last used will be the greatest common measure required; and if such a divisor cannot be found, the terms of the fraction have no common measure.

NOTE. If any of the divisors, in the course of the operation, become negative, they may have their signs changed, or be taken affirmatively, without altering the truth of the result; and if the first term of a divisor should not be exactly contained in the first term of the dividend, the several terms of the latter may be multiplied by any number, or quantity, that will render the division complete.*

EXAMPLES.

1. Required the greatest common measure of the fraction $\frac{x^4-1}{x^5+x^3}$.

$$\begin{array}{rl}
x^4-1)\,x^5+x^3\,(x & \\
x^5-x & \\
\hline
x^3+x & \\
\text{or } x^2+1 \,|\, x^4-1\,(x^2-1 & \\
x^4+x^2 & \\
\hline
-x^2-1 & \\
-x^2-1 & \\
\hline
* &
\end{array}$$

Whence x^2+1 is the greatest common measure required.

---

* In finding the greatest common measure of two quantities, either of them may be multiplied, or divided, by any number or quantity, which is not a divisor of the other, or that contains no factor, which is common to them both, without, in any respect, changing the result.

It may here, also, be further added, that the common measure, or divisor, of any number of quantities, may be determined in a similar manner to that given above, by first finding the common measure of two of them, and then of that common measure and the third; and so on to the last.

2. Required the greatest common measure of the fraction $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$

$$\begin{array}{l}
x^2+2bx+b^2)x^3-b^2x(x \\
\qquad\qquad\qquad x^3+2bx^2+b^2x \\
\qquad\qquad\qquad \overline{-2bx^2-2b^2x} \\
\qquad\qquad\qquad\quad \text{or } x+b\ \big|\ x^2+2bx+b^2(x+b \\
\qquad\qquad\qquad\qquad\qquad\quad x^2+\ bx \\
\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad bx+b^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad bx+b^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \overline{\qquad *\qquad}
\end{array}$$

Whence $x+b$ is the greatest common measure required.

3. Required the greatest common measure of the fraction $\dfrac{3a^2-2a-1}{4a^3-2a^2-3a+1}$.

$$\begin{array}{l}
3a^2-2a-1)4a^3-2a^2-3a+1. \\
\qquad\qquad\qquad 3 \\
\qquad\qquad\qquad \overline{12a^3-6a^2-9a+3(4a} \\
\qquad\qquad\qquad 12a^3-8a^2-4a \\
\qquad\qquad\qquad \overline{\qquad 2a^2-5a+3)3a^2-2a-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{6a^2-\ 4a-\ 2(3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 6a^2-15a+\ 9 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad 11a-11 \text{ or } a-1}
\end{array}$$

Whence, since $a-1)2a^2-5a+3(2a-3$, it follows that the last divisor $a-1$ is the common measure required.

In this case, the common process has been interrupted in the last step, merely to prevent the work over-running the page.

4. It is required to find the greatest common measure of $\dfrac{x^3-a^3}{x^4-a^4}$. *Ans.* $x-a$.

5. Required the greatest common measure of the fraction $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$. *Ans.* a^2-x^2.

6. Required the greatest common measure of the fraction $\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$. *Ans.* x^2+ax+a^2.

7. Required the greatest common measure of the fraction $\frac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$. *Ans.* $a-3b$.

CASE II.

To reduce Fractions to their lowest or most simple terms.

RULE.

Divide the terms of the fraction by any number, or quantity, that will divide each of them without leaving a remainder, and the result will be the fraction required. Or find their greatest common measure, as in the last rule, by which divide both the numerator and denominator, and it will give the fraction as before.

EXAMPLES.

1 Reduce $\frac{a^2bc}{5a^2b^2}$ and $\frac{x^2}{ax+x^2}$ to their lowest terms.

Here $\frac{a^2bc}{5a^2b^2}=\frac{c}{5b}$ *Ans.* And $\frac{x^2}{ax+x^2}=\frac{x}{a+x}$ *Ans.*

2. It is required to reduce $\frac{cx+x^2}{a^2c+a^2x}$ to its lowest terms.

$$\begin{array}{r|l} \text{Here } cx+x^2 & a^2c+a^2x \\ \text{or } c+x & a^2c+a^2x(a^2 \\ & \underline{a^2c+a^2x} \\ & \quad * \end{array}$$

Whence $c+x$ is the greatest common measure; and

$$c+x)\frac{cx+x^2}{a^2c+a^2x}=\frac{x}{a^2} \text{ the fraction required.}$$

3. It is required to reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}$ to its lowest terms.

$$\begin{array}{l}
x^2+2bx+b^2)\,x^3-b^2x\,(x \\
\qquad\qquad\quad\ \underline{x^3+2bx^2+b^2x} \\
\qquad\qquad\qquad -2bx^2-2b^2x \\
\qquad\qquad\qquad\quad \text{or } x+b\,|\,x^2+2bx+b^2\,(x+b \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2+bx} \\
\qquad\qquad\qquad\qquad\qquad\qquad bx+b^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{bx+b^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad *
\end{array}$$

Whence $x+b$ is the greatest common measure; and $x+b)\ \frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x^2-bx}{x+b}$ the fraction required.

And the same answer would have been found, if x^3-b^2x had been made the divisor instead of $x^2+2bx+b^2$.

4. It is required to reduce $\frac{x^4-a^4}{x^5-a^2x^3}$ to its lowest terms.

Ans. $\frac{x^2+a^2}{x^3}$.

5. It is required to reduce $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ to its lowest terms

Ans. $\frac{3a-x}{3a+x}$.

6. It is required to reduce $\frac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest terms.

Ans. $\frac{2}{3}$.

7. It is required to reduce $\frac{9x^5+2x^3+4x^2-x+1}{15x^4-2x^3+10x^2-x+2}$ to its lowest terms.

Ans. $\frac{3x^3+x^2+1}{5x^2+x+2}$.

CASE III.

To reduce a mixed Quantity to an improper Fraction.

RULE.

Multiply the integral part by the denominator of the fraction, and to the product add the numerator, when it is

affirmative, or subtract it when negative; then the result, placed over the denominator, will give the improper fraction required.

EXAMPLES.

1. Reduce $3\frac{2}{5}$ and $a-\frac{b}{c}$ to improper fractions.

Here $3\frac{2}{5}=\frac{3\times5+2}{5}=\frac{15+2}{5}=\frac{17}{5}$ *Ans.*

And $a-\frac{b}{c}=\frac{a\times c-b}{c}=\frac{ac-b}{c}$ *Ans*

2. Reduce $x+\frac{a}{x}$ and $x-\frac{a^2-x^2}{x}$ to improper fractions.

Here $x+\frac{a}{x}=\frac{x\times x+a}{x}=\frac{x^2+a}{x}$ *Ans.*

And $x-\frac{a^2-x^2}{x}=\frac{x^2-a^2+x^2}{x}=\frac{2x^2-a^2}{x}$ *Ans.*

3. Let $1-\frac{2x}{a}$ be reduced to an improper fraction.

Ans. $\frac{a-2x}{a}$.

4 Let $5a-\frac{3x-b}{a}$ be reduced to an improper fraction.

Ans. $\frac{5a^2-3x+b}{a}$.

5. Let $x-\frac{ax+x^2}{2a}$ be reduced to an improper fraction.

Ans. $\frac{ax-x^2}{2a}$.

6. Let $5+\frac{2x-7}{3x}$ be reduced to an improper fraction.

Ans. $\frac{17x-7}{3x}$.

7. Let $1-\frac{x-a-1}{a}$ be reduced to an improper fraction.

Ans. $\frac{2a-x+1}{a}$.

8. Let $1+2x-\frac{x-3}{5x}$ be reduced to an improper fraction.

Ans. $\frac{10x^2+4x+3}{5x}$.

CASE IV.

To reduce an improper Fraction to a whole or mixed quantity.

RULE

Divide the numerator by the denominator, for the integral part, and place the remainder, if any, over the denominator, for the fractional part; then the two, joined together, with the proper sign between them, will give the mixed quantity required.

EXAMPLES

1. Reduce $\frac{27}{5}$ and $\frac{ax+a^2}{x}$ to mixed quantities.

Here $\frac{27}{5}=27\div5=5\frac{2}{5}$. *Ans.*

And $\frac{ax+a^2}{x}=(ax+a^2)\div x=a+\frac{a^2}{x}$ *Ans.*

2. It is required to reduce the fraction $\frac{ax-x^2}{x}$ to a whole quantity. *Ans.* $a-x$.

3. It is required to reduce the fraction $\frac{ab-2a^2}{ab}$ to a mixed quantity. *Ans.* $1-\frac{2a}{b}$.

4. It is required to reduce the fraction $\frac{a^2+x^2}{a-x}$ to a mixed quantity. *Ans.* $a+x+\frac{2x^2}{a-x}$.

5. It is required to reduce the fraction $\frac{x^3-y^3}{x-y}$ to a whole quantity. *Ans.* x^2+xy+y^2.

6. It is required to reduce the fraction $\frac{10x^2-5x+3}{5x}$ to a mixed quantity *Ans.* $2x-1+\frac{3}{5x}$.

CASE V.

To reduce Fractions to other equivalent ones, that shall have a common denominator.

RULE.

Multiply each of the numerators, separately, by all the denominators, except its own, for the new numerators, and all the denominators together for a common denominator.*

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to fractions that shall have a common denominator.

Here $\left.\begin{array}{l} a\times c=ac \\ b\times b=b^2 \end{array}\right\}$ the new numerators.

$b\times c=bc$ the common denominator.

* It may here be remarked, that if the numerator and denominator of a fraction be either both multiplied, or both divided, by the same number, or quantity, its value will not be altered : thus,

$\frac{2}{3}=\frac{2\times3}{3\times3}=\frac{6}{9}$, and $\frac{3}{12}=\frac{3\div3}{12\div3}=\frac{1}{4}$; or $\frac{a}{b}=\frac{ac}{bc}$, and $\frac{ab}{bc}=\frac{a}{c}$;

which method is often of considerable use in reducing fractions more readily to a common denominator.

Whence $\frac{a}{b}$ and $\frac{b}{c}=\frac{ac}{bc}$ and $\frac{b^2}{bc}$, the fractions required.

The rule given above is general, and may be applied in all cases, but that which follows will often be found more expeditious.

RULE.

First, find the least common multiple of all the denominators of the given fractions, and it will be the least common denominator.

Secondly, divide this common denominator by each of the given denominators separately, and multiply the quotients by the several numerators, the products will be the new numerators.

NOTE.—If one or more of the given denominators be multiples of any of the others, reject those of which they are the multiples, then arrange the remaining denominators in one line, and divide each of them, or the greatest number of them, by any quantity that will divide them without a remainder; write the quotients and the undivided quantities in a line underneath; divide this second line as before, and so on till there are no *two* quantities that can be divided.—The products of the divisors, quotients, and undivided quantities will give the least common multiple required.

Ex. 1. Reduce $\frac{2a}{15}$, $\frac{2}{3b}$, $\frac{c}{5}$, $\frac{3}{7d}$, $\frac{4x}{21}$, and $\frac{5}{6}$ to fractions having a common denominator.

$$\begin{array}{l} 3)\underline{15,\ \cancel{3}b,\ \ \cancel{5},\ \cancel{7}d,\ 21,\ 6} \text{ the denominators.} \\ \quad \underline{5,\ \ b,\ \ ..,\ \ d,\ \ 7,\ 2} \end{array}$$

The numbers which are cancelled, *viz.* 3, 5, 7, may be rejected, because 15 is a multiple of the two former, and 21 of the latter.

Then $3\times5\times b\times d\times7\times2=210bd$, the least common denominator.

$$\left.\begin{array}{l}
\dfrac{210bd}{15}\times 2a=28abd \\
\dfrac{210bd}{3b}\times 2 \;=140d \\
\dfrac{210bd}{5}\times c \;=42bcd \\
\dfrac{210bd}{7d}\times 3 \;=90b \\
\dfrac{210bd}{21}\times 4x=40bdx \\
\dfrac{210bd}{6}\times 5 \;=175bd
\end{array}\right\}\text{the new numerators.}$$

Hence the new fractions in their lowest terms having a common denominator, are

$$\frac{28abd}{210bd},\ \frac{140d}{210bd},\ \frac{42bcd}{210bd},\ \frac{90b}{210bd},\ \frac{40bdx}{210bd},\ \frac{175bd}{210bd}$$

Ex. 2. Reduce $\frac{x}{12d}$, $\frac{3a}{14b}$, $\frac{2d}{15c}$, $\frac{3}{16a}$, $\frac{b}{18x}$, $\frac{3x}{20c}$ to equivalent fractions having a common denominator.

3)$12d$, $14b$, $15c$, $16a$, $18x$, $20c$, the denominators.

2)$4d$, $14b$, $5c$, $16a$, $6x$, $20c$ {here 4 and $5c$ may be rejected, being the multiples of $20c$.

2)d, $7b$, .., $8a$, $3x$, $10c$

d, $7b$, .., $4a$, $3x$, $5c$

Then $3\times 2\times 2\times d\times 7b\times 4a\times 3x\times 5c=5040abcdx$, the least common denominator.

$$\left.\begin{array}{l}
\dfrac{5040abcdx}{12d}\times x= 420abcx^2 \\
\dfrac{5040abcdx}{14b}\times 3a=1080a^2cdx \\
\dfrac{5040abcdx}{15c}\times 2d= 672abd^2x \\
\dfrac{5040abcdx}{16a}\times 3 = 945bcdx \\
\dfrac{5040abcdx}{18x}\times b= 280ab^2cd \\
\dfrac{5040abcdx}{20c}\times 3x= 756abdx^2
\end{array}\right\}\text{the new numerators.}$$

Hence the new fractions in their lowest terms having a common denominator, are

$$\frac{420abcx^2}{5040abcdx}, \frac{1080a^2cdx}{5040abcdx}, \frac{672abd^2x}{5040abcdx}, \frac{945bcdx}{5040abcdx},$$
$$\frac{280ab^2cd}{5040abcdx}, \frac{756abdx^2}{5040abcdx}.$$

2. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions having a common denominator. *Ans.* $\frac{2cx}{ac}$ and $\frac{ab}{ac}$.

3. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to equivalent fractions having a common denominator. *Ans.* $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

4. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d, to equivalent fractions having a common denominator. *Ans.* $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

5. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{4x}{5}$, to fractions having a common denominator. *Ans.* $\frac{45}{60}$, $\frac{40x}{60}$, and $\frac{60a+48x}{60}$

6. Reduce $\frac{a}{2}$, $\frac{3x}{7}$ and $\frac{a+x}{a-x}$, to fractions having a com mon denominator.

Ans. $\frac{7a^2-7ax}{14a-14x}$, $\frac{6ax-6x^2}{14a-14x}$, and $\frac{14a+14x}{14a-14x}$.

CASE VI.

To add fractional Quantities together.

RULE.

Reduce the fractions, if necessary, to a common denominator; then add all the numerators together, and

under their sum put the common denominator, and it will give the sum of the fractions required.*

EXAMPLES.

1. It is required to find the sum of $\frac{x}{2}$ and $\frac{x}{3}$.

Here $\left.\begin{array}{l} x\times 3=3x \\ x\times 2=2x \end{array}\right\}$ the numerators.

And $2\times 3=6$ the common denominator.

Whence $\frac{3x}{6}+\frac{2x}{6}=\frac{5x}{6}$, the sum required.

2. It is required to find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

Here $\left.\begin{array}{l} a\times d\times f=adf \\ c\times b\times f=cbf \\ e\times b\times d=ebd \end{array}\right\}$ the numerators.

And $b\times d\times f=bdf$ the common denominator.

Whence $\frac{adf}{bdf}+\frac{cbf}{bdf}+\frac{ebd}{bdf}=\frac{adf+cbf+ebd}{bdf}$ the sum required.

3. It is required to find the sum of $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$.

Here, taking only the fractional parts,

we shall have $\left\{\begin{array}{l} 3x^2\times c=3cx^2 \\ 2ax\times b=2abx \end{array}\right\}$ the numerators.

And $b\times c=bc$ the common denominator.

Whence $a-\frac{3cx^2}{bc}+b+\frac{2abx}{bc}=a+b+\frac{2abx-3cx^2}{bc}$ the sum required.

* In the adding or subtracting of mixed quantities, it is best to bring the fractional parts only to a common denominator, and then to affix their sum or difference to the sum or difference of the integral parts, interposing the proper sign.

4. It is required to find the sum of $\frac{2x}{5}$ and $\frac{5x}{7}$.

Ans. $\frac{39x}{35}$.

5. It is required to find the sum of $\frac{3x}{2a}$ and $\frac{x}{5}$.

Ans. $\frac{(15+2a)x}{10a}$.

6. It is required to find the sum of $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$.

Ans. $\frac{13x}{12}$.

7. It is required to find the sum of $\frac{4x}{7}$ and $\frac{x-2}{5}$

Ans. $\frac{27x-14}{35}$.

8. Required the sum of $2a$, $3a+\frac{2x}{5}$ and $a-\frac{8x}{9}$.

Ans. $6a-\frac{22x}{45}$.

9. Required the sum of $2a+\frac{3x}{5}$, $\frac{a}{a-x}$ and $\frac{a-x}{a}$.

Ans. $2a+\frac{10a^2-10ax+3a^2x-3ax^2+5x^2}{5a^2-5ax}$.

10. Required the sum of $5x+\frac{x-2}{3}$ and $4x-\frac{2x-3}{5x}$.

Ans. $9x+\frac{5x^2-16x+9}{15x}$.

11 It is required to find the sum of $5x$, $\frac{2a}{3x^2}$, and $\frac{a+2x}{4x}$

Ans. $5x+\frac{8ax+3ax^2+6x^3}{12x^3}$.

CASE VII.

To subtract one fractional Quantity from another.

RULE.

Reduce the fractions to a common denominator, if necessary, as in addition; then subtract the less numerator from the greater, and under the difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES.

1. It is required to find the difference of $\frac{2x}{3}$ and $\frac{3x}{5}$.

$$\text{Here}\quad \left.\begin{array}{l} 2x\times 5=10x \\ 3x\times 3=\ \ 9x \end{array}\right\}\text{ the numerators.}$$

$$\text{And}\quad 3\ \times 5=15\ \text{ the common denominator.}$$

Whence $\frac{10x}{15}-\frac{9x}{15}=\frac{x}{15}$, the difference required.

2. It is required to find the difference of $\frac{x-a}{2b}$ and $\frac{2a-4x}{3c}$.

$$\left.\begin{array}{l} (x\ \ -a\ )\times 3c=3cx-3ac \\ (2a-4x)\times 2b=4ab-8bx \end{array}\right\}\text{ the numerators.}$$

And $2b\times 3c=6bc$ the common denominator.

Whence $\frac{3cx-3ac}{6bc}-\frac{4ab-8bx}{6bc}=\frac{3cx-3ac-4ab+8bx}{6bc}$ the difference required.

3. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$. *Ans.* $\frac{39x}{35}$.

4. Required the difference of $15y$ and $\frac{1+2y}{8}$.

Ans. $\frac{118y-1}{8}$.

5. Required the difference of $\frac{ax}{b-c}$ and $\frac{ax}{b+c}$.

Ans. $\frac{2acx}{b^2-c^2}$.

6. Required the difference of $x-\frac{x-a}{c}$ and $x+\frac{x}{2b}$.

Ans. $\frac{cx+2bx-2ab}{2bc}$.

7. Required the difference of $a+\frac{a-x}{a+x}$ and $a-\frac{a+x}{a-x}$.

Ans. $\frac{2a^2+2x^2}{a^2-x^2}$.

8. It is required to find the difference of $ax+\frac{2x+7}{8}$ and $x-\frac{5x-6}{21}$.

Ans. $\frac{168ax-86x+99}{168}$.

9. It is required to find the difference of $2x+\frac{3x-5}{7}$ and $3x+\frac{11x-10}{15}$.

Ans. $\frac{137x+5}{105}$.

10. It is required to find the difference of $a+\frac{a-x}{a(a+x)}$ and $\frac{a+x}{a(a-x)}$.

Ans. $a-\frac{4x}{a^2-x^2}$.

CASE VIII.

To multiply fractional Quantities together.

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator; and the

former of these, being placed over the latter, will give the product of the fractions, as required.*

EXAMPLES.

1. It is required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

Here $\frac{x \times 2x}{6 \times 9} = \frac{2x^2}{54} = \frac{x^2}{27}$ the product required.

2. It is required to find the continued product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$.

Here $\frac{x \times 4x \times 10x}{2 \times 5 \times 21} = \frac{40x^3}{210} = \frac{4x^3}{21}$ the product required.

3. It is required to find the product of $\frac{x}{a}$ and $\frac{a+x}{a-x}$.

Here $\frac{x \times (a+x)}{a \times (a-x)} = \frac{x^2+ax}{a^2-ax}$ the product required.

4. It is required to find the product of $\frac{3x}{2}$ and $\frac{5x}{3b}$.

Ans. $\frac{5x^2}{2b}$.

5. It is required to find the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$.

Ans. $\frac{3x^3}{5a}$.

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* When the numerator of one of the fractions to be multiplied, and the denominator of the other, can be divided by some quantity which is common to each of them, the quotients may be used instead of the fractions themselves.

Also, when a fraction is to be multiplied by an integer, it is the same whether the numerator be multiplied by it, or the denominator divided by it.

Or if an integer is to be multiplied by a fraction, or a fraction by an integer, the integer may be considered as having unity for its denominator, and the two be then multiplied together in the usual manner.

6. It is required to find the continued product of $\frac{2x}{3}$, $\frac{4x^2}{7}$ and $\frac{a}{a+x}$. *Ans.* $\frac{8ax^3}{21a+21x}$.

7. It is required to find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{5ac}{2b}$. *Ans.* $15ax$.

8. It is required to find the product of $2a + \frac{bx}{a}$ and $6a - \frac{b}{ax}$. *Ans.* $12a^2+6bx-\frac{2b}{x}-\frac{b^2}{a^2}$.

9. It is required to find the continued product of $3x$, $\frac{x+1}{2a}$ and $\frac{x-1}{a+b}$. *Ans.* $\frac{3x^3-3x}{2a^2+2ab}$.

10. It is required to find the continued product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$ and $a+\frac{ax}{a-x}$. *Ans* $\frac{a^3-a^2b}{x}$.

CASE IX.

To divide one fractional Quantity by another.

RULE.

Multiply the denominator of the divisor by the numerator of the dividend, for the numerator; and the numerator of the divisor by the denominator of the dividend, for the denominator. Or, which is more convenient in practice, multiply the dividend by the reciprocal of the divisor, and the product will be the quotient required.*

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* When a fraction is to be divided by an integer, it is the same whether the numerator be divided by it, or the denominator multiplied by it.

Also, when the two numerators, or the two denominators, can be divided by some common quantity, that quantity may be thrown out of each, and the quotients used instead of the fractions first proposed.

EXAMPLES.

1. It is required to divide $\frac{x}{3}$ by $\frac{2x}{9}$.

Here $\frac{x}{3} \div \frac{2x}{9} = \frac{x}{3} \times \frac{9}{2x} = \frac{9x}{6x} = \frac{3}{2} = 1\frac{1}{2}$ *Ans.*

2. It is required to divide $\frac{2a}{b}$ by $\frac{4c}{d}$.

Here $\frac{2a}{b} \times \frac{d}{4c} = \frac{2ad}{4bc} = \frac{ad}{2bc}$ *Ans.*

3. It is required to divide $\frac{x+a}{x-b}$ by $\frac{x+b}{5x+a}$.

Here $\frac{x+a}{x-b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{x^2-b^2}$ *Ans.*

4. It is required to divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$.

Here $\frac{2x^2}{a^3+x^3} \times \frac{a+x}{x} = \frac{2x^2(a+x)}{x(a^3+x^3)} = \frac{2x^2(a+x)}{x(a^2-ax+x^2).(a+x)} = \frac{2x}{a^2-ax+x^2}$.

5. It is required to divide the fraction $\frac{7x}{5}$ by $\frac{3}{x}$.

Ans. $\frac{7x^2}{15}$.

6. It is required to divide the fraction $\frac{4x^2}{7}$ by $5x$.

Ans. $\frac{4x}{35}$.

7. It is required to divide $\frac{x+1}{6}$ by $\frac{2x}{3}$. *Ans.* $\frac{x+1}{4x}$.

8. It is required to divide $\frac{x}{1-x}$ by $\frac{x}{5}$. *Ans.* $\frac{5}{1-x}$.

9. It is required to divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$.

Ans. $\frac{2a+x}{c^2+cx+x^2}$.

10. It is required to divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

Ans. $\frac{x^2+b^2}{x}$.

INVOLUTION.

Involution is the raising of powers from any proposed root; or the method of finding the square, cube, biquadrate, &c., of any given quantity.

RULE I.

Multiply the index of the quantity by the index of the power to which it is to be raised, and the result will be the power required.

Or multiply the quantity into itself as many times less one as is denoted by the index of the power, and the last product will be the answer.

Note. When the sign of the root is $+$, all the powers of it will be $+$; and when the sign is $-$, all the even powers will be $+$, and the odd powers $-$; as is evident from multiplication.*

EXAMPLES.

a the root.	a^2 the root.
$a^2=$square.	$a^4=$square.
$a^3=$cube.	$a^6=$cube.
$a^4=$4th power.	$a^8=$4th power.
$a^5=$5th power.	$a^{10}=$5th power.
&c.	&c.
$-3a$ the root.	$-2ax^2$ the root.
$+9a^2=$square.	$+4a^2x^4=$square.
$-27a^3=$cube.	$-8a^3x^6=$cube.
$+81a^4=$4th power.	$+16a^4x^8=$4th power.
&c.	&c.

* Any power of the product of two or more quantities is equal to the same power of each of the factors multiplied together. And any power of a fraction is equal to the same power of the numerator divided by the like power of the denominator.

Also, a^m raised to the nth power is a^{mn}, and $-a^m$ raised to the nth power is $\pm a^{mn}$, according as the index n is an even or an odd number.

$\dfrac{x}{a}$ the root.

$\dfrac{x^2}{a^2}=$ square.

$\dfrac{x^3}{a^3}=$ cube.

$\dfrac{x^4}{a^4}=$ 4th power.

&c.

$-\dfrac{2ax^2}{3b}$ the root.

$+\dfrac{4a^2x^4}{9b^2}=$ square.

$-\dfrac{8a^3x^6}{27b^3}=$ cube.

$+\dfrac{16a^4x^8}{81b^4}=$ 4th power.

&c.

$$\begin{array}{llll}
x-a & & & \text{the root.} \\
x-a & & & \\
\hline
x^2-ax & & & \\
\quad -ax+a^2 & & & \\
\hline
x^2-2ax+a^2 & & & \text{square.} \\
x-a & & & \\
\hline
x^3-2ax^2+a^2x & & & \\
\quad -ax^2+2a^2x-a^3 & & & \\
\hline
x^3-3ax^2+3a^2x-a^3 & & & \text{cube.}
\end{array}$$

$$\begin{array}{llll}
x+a & & & \text{the root.} \\
x+a & & & \\
\hline
x^2+ax & & & \\
\quad +ax+a^2 & & & \\
\hline
x^2+2ax+a^2 & & & \text{square.} \\
x+a & & & \\
\hline
x^3+2ax^2+a^2x & & & \\
\quad +ax^2+2a^2x+a^3 & & & \\
\hline
x^3+3ax^2+3a^2x+a^3 & & & \text{cube.}
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the cube, or third power, of $2a^2$.

Ans. $8a^6$.

2. Required the biquadrate, or 4th power, of $2a^2x$.

Ans. $16a^8x^4$.

3. Required the cube, or third power, of $-\dfrac{2}{3}x^2y^3$.

Ans. $-\dfrac{8}{27}x^6y^9$.

4. Required the biquadrate, or 4th power, of $-\dfrac{3a^2x}{5b^2x}$

Ans. $\dfrac{81a^8x^4}{625b^8x^4}=\dfrac{81a^8}{625b^8}$.

5. Required the 4th power of $a+x$; and the 5th power of $a-y$. *Ans.* $a^4+4a^3x+6a^2x^2+4ax^3+x^4$ and $a^5-5a^4y+10a^3y^2-10a^2y^3+5ay^4-y^5$

RULE II.

A binomial or residual quantity may also be raised to any power, without the trouble of continual involution, as follows:—

1. Find the terms without the coefficients, by observing that the index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and that in the following quantity, the indices of the terms are 1, 2, 3, 4, &c.

2. To find the coefficients, observe that those of the first and last terms are always 1; and that the coefficient of the second term is the index of the power of the first: and for the rest, if the coefficient of any term be multiplied by the index of the leading quantity in it, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following; or instead of dividing the product of the coefficient and exponent, divide only one of them by the number (for one will always be divisible by it) and multiply the quotient by the other. Thus, in the following example, the first term of the required expansion is a^5, because a is the first term of the given binomial, and 5 is the proposed power; also dividing the coefficient of the term now found, which is 1, by 1, the number of terms found, the quotient is 1, which multiplied by the exponent 5 gives 5 for the coefficient of the second term, so that the complete second term is $5a^4x$; again, dividing the exponent 4 by 2, the number of terms now found, and multiplying the quotient by the coefficient 5, we find 10 for the coefficient of the next term, so that the complete third term is $10a^3x^2$. In like manner, dividing the coefficient 10 by 3, the number of terms already found, and multiplying the quotient by the exponent 3, we have 10 for the coefficient of the 4th term, and so on.

Note. The whole number of terms will be one more than the index of the given power; and when both terms of the root are $+$, all the terms of the power will be $+$; but if the second term be $-$, all the odd terms will be $+$, and the even terms $-$; or, which is the same thing, the terms will be $+$ and $-$ alternately.*

---

* The rule here laid down, which is the same in the case of integral

EXAMPLES.

1. Let $a+x$ be involved, or raised to the 5th power.

$(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$.

2. Let $a-x$ be involved, or raised to the 6th power.

Proceeding here as in the former example, we have

$(a-x)^6=a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$.

3. Required the 4th power of $a+x$, and the 5th power of $a-x$.

Ans. $(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4$, and

$(a-x)^5=a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

4. Required the 6th power of $a+x$, and the 7th power of $a-y$.

Ans. $(a+x)^6=a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6$, and $(a-y)^7=a^7-7a^6y+21a^5y^2-35a^4y^3+35a^3y^4-21a^2y^5+7ay^6-y^7$.

5. Required the 5th power of $2+x$, and the cube of $a-bx+c$.

Ans. $(2+x)^5=32+80x+80x^2+40x^3+10x^4+x^5$ and $(a-bx+c)^3=a^3+3a^2c+3ac^2+c^3-3a^2bx-6acbx-3c^2bx+3ab^2x^2+3b^2cx^2-b^3x^3$.

EVOLUTION.

Evolution, or the extraction of roots, is the reverse of involution, or the raising powers; being the method of finding the square root, cube root, &c. of any given quantity.

---

powers as the celebrated binomial theorem of Sir I. Newton, hereafter given, may be expressed in general terms, as follows:—

$$(a+b)^m=a^m+ma^{m-1}b+\frac{m(m-1)}{2}a^{m-2}b^2+\frac{m(m-1)(m-2)}{2.3}a^{m-3}b_3\ \&c$$

$$(a-b)^m=a^m-ma^{m-1}b+\frac{m(m-1)}{2}a^{m-2}b^2-\frac{m(m-1)(m-2)}{2.3}a^{m-3}b^3\ \&c.$$

which formulæ will also equally hold when m is a fraction, as will be more fully explained hereafter.

Let a and b each $=1$ in the above expansion of $(a+b)^m$, and we have $2^m=1+m+m.\frac{(m-1)}{1.2}+\frac{m.(m-1).(m-2)}{1.2.3}+\&c.=$ sum of the coefficients in the mth power.

CASE I.

To find any Root of a simple Quantity.

RULE.

Extract the root of the coefficient for the numeral part, and the root of the quantity subjoined to it for the literal part; then these, joined together, will be the root required.

And if the quantity proposed be a fraction, its root will be found by taking the root both of its numerator and denominator.

Note. The square root, the fourth root, or any other even root, of an affirmative quantity, may be either $+$ or $-$ Thus, $\sqrt{a^2}=+a$ or $-a$, and $\sqrt[4]{b^4}=+b$ or $-b$, &c. But the cube root, or any other odd root, of a quantity, will have the same sign as the quantity itself. Thus,

$$\sqrt[3]{a^3}=a;\ \sqrt[3]{-a^3}=-a;\ \sqrt[5]{-a^5}=-a, \text{ \&c.*}$$

It may here, also, be further remarked, that any even root of a negative quantity is unassignable.

Thus, $\sqrt{-a^2}$ cannot be determined, as there is no quantity, either positive or negative ($+$ or $-$), that, when multiplied by itself, will produce $-a^2$.

EXAMPLES.

1. Find the square root of $9x^2$; and the cube root of $8x^3$.

Here $\sqrt{9x^2}=\sqrt{9}\times\sqrt{x^2}=3\times x=3x$. *Ans.*
And $\sqrt[3]{8x^3}=\sqrt[3]{8}\times\sqrt[3]{x^3}=2\times x=2x$. *Ans.*

2. It is required to find the square root of $\frac{a^2x^2}{4c^2}$, and the cube root of $-\frac{8a^3x^3}{27c^3}$.

---

* The reason why $+a$ and $-a$ are each the square root of a^2 is obvious, since, by the rule of multiplication, $(+a)\times(+a)$ and $(-a)\times(-a)$ are both equal to a^2.

And for the cube root, fifth root, &c., of a negative quantity, it is plain, from the same rule, that

$$(-a)\times(-a)\times(-a)=-a^3;\text{ and }(-a^3)\times(+a^2)=-a^5.$$

And consequently $\sqrt[3]{-a^3}=-a$, and $\sqrt[5]{-a^5}=-a$.

Here $\sqrt{\dfrac{a^2x^2}{4c^2}}=\dfrac{\sqrt{a^2x^2}}{\sqrt{4c^2}}=\dfrac{ax}{2c}$; and $\sqrt[3]{-\dfrac{8a^3x^3}{27c^3}}=-\dfrac{2ax}{3c}$.

3. It is required to find the square root of $4a^2x^6$.

Ans. $2ax^3$.

4. It is required to find the cube root of $-125a^3x^6$.

Ans. $-5ax^2$.

5. It is required to find the 4th root of $256a^4x^8$.

Ans. $4ax^2$.

6. It is required to find the square root of $\dfrac{4a^4}{9x^2y^2}$.

Ans. $\dfrac{2a^2}{3xy}$.

7. It is required to find the cube root of $\dfrac{8a^3}{125x^6}$.

Ans. $\dfrac{2a}{5x^2}$.

8. It is required to find the 5th root of $-\dfrac{32a^5x^{10}}{243}$.

Ans. $\dfrac{-2ax^2}{3}$.

CASE II.

To extract the square Root of a compound Quantity.

RULE.

1. Range the terms, of which the quantity is composed, according to the dimensions of some letter in them, beginning with the highest, and set the root of the first term in the quotient.

2. Subtract the square of the root, thus found, from the first term, and bring down the two next terms to the remainder, for a dividend.

3. Divide this dividend by double that part of the root already determined, and set the result both in the quotient and divisor.

4. Multiply the divisor, so increased, by the term of the root last placed in the quotient, and subtract the product from the dividend; and so on, as in common arithmetic.

EXAMPLES.

1. Extract the square root of $x^4-4x^3+6x^2-4x+1$.

$$\begin{array}{l}
\qquad\qquad x^4-4x^3+6x^2-4x+1(x^2-2x+1 \\
\qquad\qquad x^4 \\
\hline
2x^2-2x)-4x^3+6x^2 \\
\qquad\quad -4x^3+4x^2 \\
\hline
\qquad 2x^2-4x+1)2x^2-4x+1 \\
\qquad\qquad\qquad\quad 2x^2-4x+1 \\
\hline
\qquad\qquad\qquad\qquad *
\end{array}$$

Ans. x^2-2x+1, the root required.

2. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

$$\begin{array}{l}
\qquad 4a^4+12a^3x+13a^2x^2+6ax^3+x^4(2a^2+3ax+x^2 \\
\qquad 4a^4 \\
\hline
4a^2+3ax)\ 12a^3x+13a^2x^2 \\
\qquad\qquad\quad 12a^3x+\ 9a^2x^2 \\
\hline
\qquad 4a^2+\ 6ax+x^2)4a^2x^2+6ax^3+x^4 \\
\qquad\qquad\qquad\qquad\quad 4a^2x^2+6ax^3+x^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad *
\end{array}$$

Note.—When the quantity to be extracted has no exact root, the operation may be carried on as far as is thought necessary, or till the regularity of the terms shows the law by which the series would be continued.

EXAMPLE.

1. It is required to extract the square root of $1+x$

$$1+x(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}\ \&c.$$

$$1$$

$$\overline{\quad\quad\quad\quad}$$

$$2+\frac{x}{2}\Big)\,x$$

$$x+\frac{x^2}{4}$$

$$\overline{\quad\quad\quad\quad}$$

$$2+x-\frac{x^2}{8}\Big)-\frac{x^2}{4}$$

$$-\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64}$$

$$\overline{\quad\quad\quad\quad}$$

$$2+x-\frac{x^2}{4}+\frac{x^3}{16}\Big)\frac{x^3}{8}-\frac{x^4}{64}$$

$$\frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256}$$

$$\overline{\quad\quad\quad\quad}$$

$$-\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}.$$

Here, if the numerators and denominators of the two last terms be each multiplied by 3, which will not alter their values, the root will become

$$1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}+\frac{3.5.7x^5}{2.4.6.8.10}\ \&c.$$

where the law of the series is manifest.

EXAMPLES FOR PRACTICE.

2. It is required to find the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$. *Ans.* $a^2+2ax+x^2$.

3. It is required to find the square root of $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}$. *Ans.* $x^2-x+\frac{1}{4}$.

4. It is required to find the square root of $4x^6-4x^4+12x^3+x^2-6x+9$. *Ans.* $2x^3-x+3$.

5. Required the square root of $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$. *Ans.* x^3+2x^2+3x+4.

6. It is required to extract the square root of a^2+b.

Ans. $a+\frac{b}{2a}-\frac{b^2}{8a^3}+$ &c.

7. It is required to extract the square root of 2, or of $1+1$. *Ans.* $1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-$&c.

CASE III.

To find any Root of a compound Quantity.

RULE.

Find the root of the first term, which place in the quotient; and having subtracted its corresponding power from that term, bring down the second term for a dividend.

Divide this by twice the part of the root above determined, for the square root; by three times the square of it, for the cube root, and so on; and the quotient will be the next term of the root.

Involve the whole of the root, thus found, to its proper power, which subtract from the given quantity, and divide the first term of the remainder by the same divisor as before; and proceed in this manner till the whole is finished.*

---

* When the root to be extracted consists of many terms, the following will be found a more convenient rule than that given in the text.

Arrange the expression as above, and if any of the powers of the leading quantity are wanted, supply them, and put zero for their coefficients; find the roots of the first and last terms, and set them down as the first and last terms of the whole root; then, if the root to be found is the nth, raise each of these quantities to the $(n-1$th) power, and multiply them by n; the second term and the last but one of the given quantity, being respectively divided by the results, will give the second term and the last but one of the whole root; if this last contains any more terms, they must be put down in their proper places, with the letters a, b, c, &c., for coefficients, and the whole expression being involved to the proper power, the quantities a, b, c, &c., will be determined by comparing the result with the given expression; but this part of the operation requires a greater proficiency in algebra than is supposed in the text.

Example.—Required the square root of

$$x^6+6x^5+13x^4+20x^3+28x^2+16x+16.$$

Here the square roots of the first and last terms are x^3 and 4, and as n is equal to 2, and $n-1$ to 1, the divisors are $2x^3$ and 8; whence $6x^5 \div 2x^3 = 3x^2$ and $16x \div 8 = 2x$, are the second term, and the last but one of the root; wherefore x^3+3x^2+2x+4 is the root required.

EXAMPLES.

1. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$
\begin{array}{rl}
 & a^4-2a^3x+3a^2x^2-2ax^3+x^4(a^2-ax+x^2 \\
 & a^4 \\
\hline
 & 2a^2)-2a^3x \\
\hline
(a^2-ax)^2= & a^4-2a^3x+a^2x^2 \\
\hline
 & \quad 2a^2)2a^2x^2= \text{ difference} \\
\hline
(x^2-ax+x^2)^2= & a^4-2a^3x+3a^2x^2-2ax^3+x^4 \\
\hline
 & *
\end{array}
$$

2. Required the cube root of $x^6+6x^5-40x^3+96x-64$.

$$
\begin{array}{rl}
 & x^6+6x^5-40x^3+96x-64(x^2+2x-4 \\
 & x^6 \\
\hline
 & 3x^4) \ 6x^5 \\
\hline
(x^2+2x)^3= & x^6+6x^5+12x^4+8x^3 \\
\hline
 & \quad 3x^4)-12x^4= \text{ difference} \\
\hline
(x^2+2x-4)^3= & x^6+6x^5-40x^3+96x-64 \\
\hline
 & *
\end{array}
$$

Whence x^2+2x-4 is the root required.

3. Required the square root of $4a^2-12ax+9x^2$.
Ans. $2a^2-3x$.

4. Required the square root of $a^2+2ab+2ac+b^2+2bc+c^2$.
Ans. $a+b+c$.

5. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
Ans. x^2-2x+1.

6. Required the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$.
Ans. $2a-3x$.

7. Required the 5th root of $32x^5-80x^4+80x^3-40x^2+10x-1$.
Ans. $2x-1$.

OF IRRATIONAL QUANTITIES, OR SURDS.

IRRATIONAL quantities, or surds, are such as have no exact root, being usually expressed by means of the radical sign, or by fractional indices; in which latter case the numerator shows the power the quantity is to be raised to, and the denominator its root.

Thus, $\sqrt{2}$ or $2^{\frac{1}{2}}$, is the square root of 2; also $\sqrt[3]{a^2}$ and $\sqrt{a^3}$, or $a^{\frac{2}{3}}$ and $a^{\frac{3}{2}}$, are, respectively, the square of the cube root of a, and the cube of the square root of a, also $a^{\frac{m}{n}}$ is the mth power of the nth root of a.*

CASE I.

To reduce a rational Quantity to the form of a Surd.

RULE.

Raise the quantity to a power corresponding with that denoted by the index of the surd; and over this new quantity place the radical sign, or proper index, and it will be of the form required.

EXAMPLES.

1. Let 3 be reduced to the form of the square root.
   Here $3\times3=3^2=9$; whence $\sqrt{9}$ *Ans.*
2. Reduce $2x^2$ to the form of the cube root.
   Here $(2x^2)^3=8x^6$; whence $\sqrt[3]{8x^6}$, or $(8x^6)^{\frac{1}{3}}$ *Ans.*
3. Let 5 be reduced to the form of the square root.
   Ans. $\sqrt{25}$.

---

* A quantity of the kind here mentioned, as for instance $\sqrt{2}$, is called an irrational number or a surd, because no number, either whole or fractional, can be found, which, when multiplied by itself, will produce 2. But its approximate value may be determined to any degree of exactness, by the common rule for extracting the square root, being 1, and certain non periodic decimals, which never terminate.

4. Let $-3x$ be reduced to the form of the cube root.

Ans. $\sqrt[3]{-27x^3}$.

5. Let $-2x$ be reduced to the form of the fourth root.

Ans. $\sqrt[4]{16x^4}$.

6. Let a^2 be reduced to the form of the fifth root; and $\sqrt{a}+\sqrt{b}$, $\frac{\sqrt{a}}{2a}$ and $\frac{a}{b\sqrt{a}}$ to the form of the square root.

Ans. $\sqrt[5]{a^{10}}$; also $\sqrt{(a+b+2\sqrt{ab})}$, $\sqrt{\frac{1}{4a}}$ and $\sqrt{\frac{a}{b^2}}$.

Note. Any rational quantity may be reduced, by the above rule, to the form of the surd to which it is joined, and their product be then placed under the same index, or radical sign.

EXAMPLES.

Thus $2\sqrt{2}=\sqrt{4}\times\sqrt{2}=\sqrt{(4\times2)}=\sqrt{8}$
And $2\sqrt[3]{4}=\sqrt[3]{8}\times\sqrt[3]{4}=\sqrt[3]{(8\times4)}=\sqrt[3]{32}$
Also $3\sqrt{a}=\sqrt{9}\times\sqrt{a}=\sqrt{(9\times a)}=\sqrt{9a}$
And $\frac{1}{2}\sqrt[3]{4a}=\sqrt[3]{\frac{1}{8}}\times\sqrt[3]{4a}=\sqrt[3]{(\frac{1}{8}\times4a)}=\sqrt[3]{\frac{a}{2}}$

1. Let $5\sqrt{6}$ be reduced to a simple radical form.

Ans. $\sqrt{150}$.

2. Let $\frac{1}{5}\sqrt{5a}$ be reduced to a simple radical form

Ans. $\sqrt{\frac{a}{5}}$.

3. Let $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}$ be reduced to a simple radical form.

Ans. $\sqrt[3]{\frac{2a}{3}}$.

CASE II.

To reduce Quantities of different Indices to others that shall have a given Index.

RULE.

Divide the indices of the proposed quantities by the

given index, and the quotients will be the new indices for those quantities.

Then, over the said quantities, with their new indices, place the given index, and they will be the equivalent quantities required.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to quantities that shall have the index $\frac{1}{6}$.

$$\text{Here } \frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times \frac{6}{1} = \frac{6}{2} = 3, \text{ the 1st index;}$$

$$\text{And } \frac{1}{3} \div \frac{1}{6} = \frac{1}{3} \times \frac{6}{1} = \frac{6}{3} = 2, \text{ the 2nd index.}$$

Whence $(3^3)^{\frac{1}{6}}$ and $(2^2)^{\frac{1}{6}}$, or $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$, are the quantities required.

2. Reduce $5^{\frac{1}{2}}$ and $6^{\frac{1}{3}}$ to quantities that shall have the common index $\frac{1}{6}$. *Ans.* $125^{\frac{1}{6}}$ and $36^{\frac{1}{6}}$.

3. Let $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$ be reduced to quantities that shall have the common index $\frac{1}{8}$. *Ans.* $16^{\frac{1}{8}}$ and $16^{\frac{1}{8}}$.

4. Let a^2 and $a^{\frac{1}{2}}$ be reduced to quantities that shall have the common index $\frac{1}{4}$. *Ans.* $(a^8)^{\frac{1}{4}}$ and $(a^2)^{\frac{1}{4}}$.

5. Let $a^{\frac{1}{2}}$ and b be reduced to quantities that shall have the common index $\frac{1}{8}$. *Ans.* $(a^4)^{\frac{1}{8}}$ and $(b^8)^{\frac{1}{8}}$.

Note. Surds may also be brought to a common index, by reducing the indices of the quantities to a common denominator, and then involving each of them to the power denoted by its numerator.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to quantities having a common index.

Here $3^{\frac{1}{2}} = 3^{\frac{3}{6}} = (3^3)^{\frac{1}{6}} = (27)^{\frac{1}{6}}$

And $4^{\frac{1}{3}} = 4^{\frac{2}{6}} = (4^2)^{\frac{1}{6}} = (16)^{\frac{1}{6}}$

Whence $(27)^{\frac{1}{6}}$ and $(16)^{\frac{1}{6}}$. *Ans.*

2. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to quantities that shall have a common index. *Ans.* $(4^4)^{\frac{1}{12}}$ and $(5^3)^{\frac{1}{12}}$.

3. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ to quantities that shall have a common index. *Ans.* $(a^3)^{\frac{1}{6}}$ and $(a^2)^{\frac{1}{6}}$.

4. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to quantities that shall have a common index. *Ans.* $(a^4)^{\frac{1}{12}}$ and $(b^3)^{\frac{1}{12}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to quantities that shall have a common index. *Ans.* $(a^m)^{\frac{1}{mn}}$ and $(b^n)^{\frac{1}{mn}}$.

CASE III.

To reduce Surds to their most simple forms.

RULE.

Resolve the given number, or quantity, into two factors, one of which shall be the greatest power contained in it, and set the root of this power before the remaining part, with the proper radical sign between them.*

* When the given surd involves a high number, this is most readily effected by a table of factors, from which it will immediately appear whether the surd admits of a decomposition of the kind required; if it does not admit of such a decomposition, it is already in its most simple form. Or when the nth foot of a large number is required, proceed thus:—Divide the quantity under the radical successively by the nth power of such numbers as will leave no remainder, and the last quotient will be the surd; then multiply the coefficient of the surd (which, when the surd stands alone, is understood to be unity) by the said numbers for the new coefficient, to which annex the last quotient under the surd character, and the simplest form will be expressed, as in the two examples following.

EXAMPLES.

1. Let $\sqrt{48}$ be reduced to its most simple form.
   Here $\sqrt{48}=\sqrt{(16\times 3)}=4\sqrt{3}$. *Ans.*
2. Let $\sqrt[3]{108}$ be reduced to its most simple form.
   Here $\sqrt[3]{108}=\sqrt[3]{(27\times 4)}=3\sqrt[3]{4}$. *Ans.*

Note 1. When any number, or quantity, is prefixed to the surd, that quantity must be multiplied by the root of the factor above mentioned, and the product joined to the other part, as before.

EXAMPLES.

1. Let $2\sqrt{32}$ be reduced to its most simple form.
   Here $2\sqrt{32}=2\sqrt{(16\times 2)}=2\times 4\sqrt{2}=8\sqrt{2}$. *Ans.*
2. Let $5\sqrt[3]{24}$ be reduced to its most simple form.
   Here $5\sqrt[3]{24}=5\sqrt[3]{(8\times 3)}=5\times 2\sqrt[3]{3}=10\sqrt[3]{3}$. *Ans.*

Note 2. A fractional surd may also be reduced to a more convenient form, by multiplying both the numerator and denominator by such a number, or quantity, as will

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Example 1. Reduce $\sqrt[3]{58320}$ to its simplest form (the nth power being the cube).

$$\begin{array}{ll} 2^3 \ldots\ldots & 8)58320 \\ 3^3 \ldots \left\{ \begin{array}{l} \\ \\ \end{array}\right. & \begin{array}{r} 9)7290 \\ 3)810 \end{array} \\ 3^3 \ldots \left\{ \begin{array}{l} \\ \\ \end{array}\right. & \begin{array}{r} 9)270 \\ 3)30 \end{array} \\ & \quad\ 10 \end{array}$$

$$\therefore \sqrt[3]{58320}=\sqrt[3]{(2^3\times 3^3\times 3^3\times 10)}= 2\times 3\times 3\sqrt[3]{10}=18\sqrt[3]{10} \quad \textit{Ans.}$$

Example 2. Reduce $7\sqrt[5]{746496}$ to its simplest form (the nth being the 5th power).

$$\begin{array}{ll} 2^5 \ldots \left\{ \begin{array}{l} \\ \\ \end{array}\right. & \begin{array}{r} 8)746496 \\ 4)93312 \end{array} \\ 6^5 \ldots\ldots \left\{ \begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right. & \begin{array}{r} 6)23328 \\ 6)3888 \\ 6)648 \\ 6)108 \\ 6)18 \end{array} \\ & \qquad\quad 3 \end{array}$$

$$\therefore 7\sqrt[5]{746496}=7\sqrt[5]{(2^5\times 6^5\times 3)}= 7\times 2\times 6\sqrt[5]{3}=84\sqrt[5]{3} \quad \textit{Ans.}$$

make the denominator a complete power of the kind required; and then joining its root, with 1 put over it, as a numerator, to the other part of the surd.* Or the operation may be more readily performed by the following rule.

RULE. *First*, divide the coefficient of the surd by the surd's denominator, and the proper coefficient will be obtained.

Secondly, multiply the numerator of the surd by its denominator raised to the (index -1)*th* power, and place the product under the radical sign; † this result, with the coefficient prefixed, will be the form required.

EXAMPLES.

1. Let $\sqrt{\frac{2}{7}}$ be reduced to its most simple form.

Here $\sqrt{\frac{2}{7}}=\frac{1}{7}\sqrt{(2\times 7)}=\frac{1}{7}\sqrt{14}$ *Ans.*

2. Let $3\sqrt[3]{\frac{2}{5}}$ be reduced to its most simple form.

Here $3\sqrt[3]{\frac{2}{5}}=\frac{3}{5}\sqrt[3]{(2\times 5^2)}=\frac{3}{5}\sqrt[3]{50}$ *Ans.*

EXAMPLES FOR PRACTICE.

3. Let $\sqrt{125}$ be reduced to its most simple form.

Ans. $5\sqrt{5}$.

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* The utility of reducing surds to their most simple forms, in order to have the answer in decimals, will be readily perceived from considering the first question above given, where it is found that $\sqrt{\frac{2}{7}}=\frac{1}{7}\sqrt{14}$; in which case it is only necessary to extract the square root of the whole number 14, or to find it in some of the tables that have been calculated for this purpose, and then divide it by 7; whereas we must otherwise have first divided the numerator by the denominator, and then have found the root of the quotient; or else have determined the root both of the numerator and denominator, and then divided the one by the other; which are each of them troublesome processes when performed by the common rules; and in the next example, for the cube root, the labour would be much greater.

† If this product should contain any *n*th powers, it may be further simplified by reducing it as in the following example. Reduce $2\sqrt[3]{\frac{3}{4}}$ to its simplest form.

$$2\sqrt[3]{\frac{3}{4}}=\frac{2}{4}\sqrt[3]{(3\times 4^2)}=\frac{1}{2}\sqrt[3]{48}=\frac{1}{2}\sqrt[3]{(6\times 2^3)}=\sqrt[3]{6} \textit{ Ans.}$$

4. Let $\sqrt{294}$ be reduced to its most simple form.
Ans. $7\sqrt{6}$.

5. Let $\sqrt[3]{56}$ be reduced to its most simple form.
Ans. $2\sqrt[3]{7}$.

6. Let $\sqrt[3]{192}$ be reduced to its most simple form.
Ans. $4\sqrt[3]{3}$.

7. Let $7\sqrt{80}$ be reduced to its most simple form.
Ans. $28\sqrt{5}$

8. Let $9\sqrt[3]{81}$ be reduced to its most simple form.
Ans. $27\sqrt[3]{3}$.

9. Let $\frac{3}{121}\sqrt{\frac{5}{6}}$ be reduced to its most simple form.

Ans. $\frac{1}{242}\sqrt{30}$.

10. Let $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$ be reduced to its most simple form.

Ans. $\frac{1}{7}\sqrt[3]{12}$.

11. Let $\sqrt{98a^2x}$ be reduced to its most simple form.
Ans. $7a\sqrt{2x}$.

12. Let $\sqrt{(x^3-a^2x^2)}$ be reduced to its most simple form.
Ans. $x\sqrt{(x-a^2)}$.

CASE IV.

To add Surd Quantities together.

RULE.

When the surds are of the same kind, reduce them to their simplest forms, as in the last case; then, if the surd part be the same in them all, annex it to the sum of the rational parts, and it will give the whole sum required.

But if the quantities have different indices, or the surd part be not the same in each of them, they can only be added together by the signs $+$ and $-$.

EXAMPLES.

1. It is required to find the sum of $\sqrt{27}$ and $\sqrt{48}$.

Here $\sqrt{27}=\sqrt{(9\times3)}=3\sqrt{3}$
And $\sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}$

Whence $7\sqrt{3}$ the sum.

2. It is required to find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

Here $\sqrt[3]{500}=\sqrt[3]{(125\times 4)}=5\sqrt[3]{4}$

And $\sqrt[3]{108}=\sqrt[3]{(27\times 4)}=3\sqrt[3]{4}$

Whence $8\sqrt[3]{4}$ the sum.

3. It is required to find the sum of $4\sqrt{147}$ and $3\sqrt{75}$.

Here $4\sqrt{147}=4\sqrt{(49\times 3)}=28\sqrt{3}$

And $3\sqrt{75}=3\sqrt{(25\times 3)}=15\sqrt{3}$

Whence $43\sqrt{3}$ the sum.

4. It is required to find the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

Here $3\sqrt{\frac{2}{5}}=\frac{3}{5}\sqrt{(5\times 2)}=\frac{3}{5}\sqrt{10}$

And $2\sqrt{\frac{1}{10}}=\frac{2}{10}\sqrt{(10\times 1)}=\frac{1}{5}\sqrt{10}$

Whence $\frac{4}{5}\sqrt{10}$ the sum.

EXAMPLES FOR PRACTICE.

5. It is required to find the sum of $\sqrt{72}$ and $\sqrt{128}$.
Ans. $14\sqrt{2}$.

6. It is required to find the sum of $\sqrt{180}$ and $\sqrt{405}$.
Ans. $15\sqrt{5}$.

7. It is required to find the sum of $3\sqrt[3]{40}$ and $\sqrt[3]{135}$.
Ans. $9\sqrt[3]{5}$.

8. It is required to find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$.
Ans. $32\sqrt[3]{2}$.

9. It is required to find the sum of $9\sqrt{243}$ and $10\sqrt{363}$.
Ans. $191\sqrt{3}$.

10. It is required to find the sum of $3\sqrt{\frac{2}{3}}$ and $7\sqrt{\frac{27}{50}}$.
Ans. $\frac{31}{10}\sqrt{6}$.

11. It is required to find the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$.
Ans. $6\frac{3}{4}\sqrt[3]{2}$.

12. It is required to find the sum of $\frac{1}{2}\sqrt{a^2b}$ and $\frac{1}{3}\sqrt{4bx^4}$.

Ans. $\left(\frac{1}{2}a+\frac{2}{3}x^2\right)\sqrt{b}$.

CASE V.

To find the Difference of Surd Quantities.

RULE.

When the surds are of the same kind, prepare the quantities as in the last rule; then the difference of the rational parts annexed to the common surd will give the whole difference required.

But if the quantities have different indices, or the surd part be not the same in each of them, they can only be subtracted by means of the sign $-$.

1. It is required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

Here $\sqrt{448}=\sqrt{(64\times7)}=8\sqrt{7}$
And $\sqrt{112}=\sqrt{(16\times7)}=4\sqrt{7}$

Whence $4\sqrt{7}$ the difference.

2. It is required to find the difference of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

Here $\sqrt[3]{192}=\sqrt[3]{(64\times3)}=4\sqrt[3]{3}$
And $\sqrt[3]{24}=\sqrt[3]{(8\times3)}=2\sqrt[3]{3}$

Whence $2\sqrt[3]{3}$ the difference.

3. It is required to find the difference of $5\sqrt{20}$ and $3\sqrt{45}$.

Here $5\sqrt{20}=5\sqrt{(4\times5)}=10\sqrt{5}$
And $3\sqrt{45}=3\sqrt{(9\times5)}=9\sqrt{5}$

Whence $\sqrt{5}$ the difference.

4. It is required to find the difference of $\frac{3}{4}\sqrt{\frac{2}{3}}$ and $\frac{2}{5}\sqrt{\frac{1}{6}}$.

Here $\frac{3}{4}\sqrt{\frac{2}{3}}=\frac{3}{4}\sqrt{\frac{6}{9}}=\frac{3}{12}\sqrt{6}=\frac{1}{4}\sqrt{6}$

And $\frac{2}{5}\sqrt{\frac{1}{6}}=\frac{2}{5}\sqrt{\frac{6}{36}}=\frac{2}{30}\sqrt{6}=\frac{1}{15}\sqrt{6}$

Whence $\frac{11}{60}\sqrt{6}$ the difference.

EXAMPLES FOR PRACTICE.

1. It is required to find the difference of $2\sqrt{50}$ and $\sqrt{18}$.
Ans. $7\sqrt{2}$.
2. It is required to find the difference of $\sqrt[3]{320}$ and $\sqrt[3]{40}$.
Ans. $2\sqrt[3]{5}$.
3. It is required to find the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.
Ans. $\frac{4}{45}\sqrt{15}$.
4. It is required to find the difference of $2\sqrt{\frac{1}{2}}$ and $\sqrt{8}$.
Ans. $\sqrt{2}$.
5. It is required to find the difference of $3\sqrt[3]{\frac{1}{3}}$ and $\sqrt[3]{72}$.
Ans. $\sqrt[3]{9}$.
6. It is required to find the difference of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$.
Ans. $\frac{1}{12}\sqrt[3]{18}$.
7. It is required to find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$. *Ans.* $(4a^2-2ax)\sqrt{5x}$.
8. It is required to find the difference of $8\sqrt[3]{a^3b}$ and $2\sqrt[3]{a^6b}$. *Ans.* $(8a-2a^2)\sqrt[3]{b}$.

CASE VI.

To multiply Surd Quantities together.

RULE.

When the surds are of the same kind, find the product of the rational parts, and the product of the surds, and the two joined together, with their common radical sign between

them, will give the whole product required; which may be reduced to its most simple form by Case III.

But if the surds are of different kinds, they must be reduced to a common index, and then multiplied together as usual.

It is also to be observed, as before mentioned, that the product of different powers, or roots of the same quantity, is found by adding their indices.

EXAMPLES.

1. It is required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here $3\sqrt{8}$

Multiplied $2\sqrt{6}$

Gives $6\sqrt{48}=6\sqrt{(16\times3)}=24\sqrt{3}$. *Ans.*

2. It is required to find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.

Here $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$

Multiplied $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$

Gives $\frac{3}{8}\sqrt[3]{\frac{10}{18}}=\frac{3}{8}\sqrt[3]{\frac{5}{9}}=\frac{3}{8}\sqrt[3]{\frac{15}{27}}=\frac{1}{8}\sqrt[3]{15}$.

3. It is required to find the product of $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$.

Here $2^{\frac{1}{2}}=2^{\frac{3}{6}}=(2^3)^{\frac{1}{6}}=8^{\frac{1}{6}}$

And $3^{\frac{1}{3}}=3^{\frac{2}{6}}=(3^2)^{\frac{1}{6}}=9^{\frac{1}{6}}$

Whence $(72)^{\frac{1}{6}}$. *Ans.*

4. It is required to find the product of $5\sqrt{a}$ and $3\sqrt[3]{a}$.

Here $5\sqrt{a}=5a^{\frac{1}{2}}=5a^{\frac{3}{6}}$

And $3\sqrt[3]{a}=3a^{\frac{1}{3}}=3a^{\frac{2}{6}}$

Whence $15a^{\frac{5}{6}}=15\,(a^5)^{\frac{1}{6}}$ or $15\sqrt[6]{a^5}$. *Ans.*

EXAMPLES FOR PRACTICE.

5. It is required to find the product of $5\sqrt[3]{8}$ and $3\sqrt[3]{5}$.
Ans. $30\sqrt[3]{5}$.

6. It is required to find the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$.
Ans. $10\sqrt[3]{9}$.

7. Required the product of $\frac{1}{4}\sqrt[3]{6}$ and $\frac{2}{15}\sqrt[3]{9}$.
Ans. $\frac{1}{10}\sqrt[3]{2}$.

8. Required the product of $\frac{1}{2}\sqrt[3]{18}$ and $5\sqrt[3]{20}$.
Ans. $5\sqrt[3]{45}$.

9. Required the product of $2\sqrt{3}$ and $13\frac{1}{2}\sqrt[3]{5}$.
Ans. $27\sqrt[6]{675}$.

10. Required the product of $72\frac{1}{4}\sqrt[3]{a^2}$ and $120\frac{1}{2}\sqrt[6]{a}$.
Ans. $8706\frac{1}{8}a^{\frac{1}{1}\frac{1}{2}}$.

11. Required the product of $4+2\sqrt{2}$ and $2-\sqrt{2}$.
Ans. 4.

12. Required the product of $(a+b)^{\frac{1}{n}}$ and $(a+b)^{\frac{1}{m}}$.
Ans. $(a+b)^{\frac{m+n}{mn}}$.

CASE VII.

To divide one Surd Quantity by another.

RULE.

When the surds are of the same kind, find the quotient of the rational parts, and the quotient of the surds, and the two joined together, with their common radical sign between them, will give the whole quotient required.

But if the surds are of different kinds, they must be reduced to a common index, and then be divided as before.

It is also to be observed, that the quotient of different powers or roots of the same quantity is found by subtracting their indices.

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

Here $\frac{8\sqrt{108}}{2\sqrt{6}}=4\sqrt{18}=4\sqrt{(9\times2)}=12\sqrt{2}$. *Ans*

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

Here $\frac{8\sqrt[3]{512}}{4\sqrt[3]{2}}=2\sqrt[3]{256}=2\sqrt[3]{(64\times4)}=8\sqrt[3]{4}$. *Ans.*

3. It is required to divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$.

Here $\frac{\frac{1}{2}\sqrt{5}}{\frac{1}{3}\sqrt{2}}=\frac{3}{2}\sqrt{\frac{5}{2}}=\frac{3}{2}\sqrt{\frac{10}{4}}=\frac{3}{4}\sqrt{10}$. *Ans.*

4. It is required to divide $\sqrt{7}$ by $\sqrt[3]{7}$.

Here $\frac{\sqrt{7}}{\sqrt[3]{7}}=\frac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}}=\frac{7^{\frac{3}{6}}}{7^{\frac{2}{6}}}=7^{\frac{3}{6}-\frac{2}{6}}=7^{\frac{1}{6}}$. *Ans.*

5. It is required to divide $6\sqrt{54}$ by $3\sqrt{2}$. *Ans.* $6\sqrt{3}$.

6. It is required to divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$. *Ans.* $2\sqrt[3]{4}$.

7. It is required to divide $5\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$.

Ans. $\frac{23}{24}\sqrt{3}$.

8 It is required to divide $3\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ by $2\frac{2}{5}\sqrt[3]{\frac{3}{4}}$

Ans. $\frac{65}{63}\sqrt[3]{3}$.

9. It is required to divide $4\frac{1}{2}\sqrt{a}$ by $2\frac{2}{3}\sqrt[3]{ab}$.

Ans. $\frac{27}{16}\left(\frac{a}{b^2}\right)^{\frac{1}{6}}$.

10. It is required to divide $32\frac{2}{5}\sqrt{a}$ by $13\frac{3}{4}\sqrt[3]{a}$.

Ans. $\frac{648}{275}a^{\frac{1}{6}}$.

11. It is required to divide $9\frac{3}{8}a^{\frac{1}{n}}$ by $4\frac{9}{11}a^{\frac{1}{m}}$.

Ans. $\frac{825}{424}a^{\frac{m-n}{mn}}$.

12. Let $\sqrt{20} + \sqrt{12}$ be divided by $\sqrt{5} - \sqrt{3}$.

Ans. $8+2\sqrt{15}$.

Note. Since the division of surds is performed by subtracting their indices, it is evident that the denominator of any fraction may be taken into the numerator, or the numerator into the denominator, by changing the sign of its index.

Also, since $\frac{a^m}{a^m} = 1$, or $= a^{m-m} = a^0$, it follows that the expression a^0 is a symbol equivalent to unity, and consequently, that it may be always replaced by 1 whenever it occurs.*

EXAMPLES.

1. Thus, $\frac{1}{a} = \frac{a^{-1}}{1}$ or a^{-1}; and $\frac{1}{a^n} = \frac{a^{-n}}{1}$ or a^{-n}.

To what is above said, we may also further observe,

1. That 0 added to or subtracted from any quantity makes it neither greater nor less; that is,

$$a + 0 = a, \text{ and } a - 0 = a.$$

2. Also, if nought be multiplied or divided by any quantity, both the product and quotient will be nought; because any number of times 0, or any part of 0, is 0; that is,

$$0 \times a, \text{ or } a \times 0 = 0, \text{ and } \frac{0}{a} = 0.$$

3. From this it likewise follows that nought divided by nought, may be a finite quantity.

For since $0 \times a = 0$, or $0 = 0 \times a$, it is evident that $\frac{0}{0} = a$.

4. Further, if any finite quantity be divided by 0, the quotient will be infinite; and if it be divided by an infinite quantity, the quotient will be 0.

For let $\frac{a}{b} = q$; then, if a be supposed to remain constant, it is plain, the less b is, the greater will be the quotient q; whence, if b be indefinitely small, q will be indefinitely great; and, consequently, when b is 0, the quotient q will be infinite: that is,

$$\frac{a}{0} = \infty, \text{ and } \frac{a}{\infty} = 0.$$

Which properties are of frequent occurrence in some of the higher parts of the science, and should be carefully remembered.

2. Also, $\frac{b}{a^2}=\frac{ba^{-2}}{1}$ or ba^{-2}; and $\frac{a^{-n}}{b^{-m}}=\frac{b^m}{a^n}$.

3. Let $\frac{1}{a^2}$ be expressed with a negative index.

Ans. a^{-2}.

4. Let $a^{-\frac{1}{2}}$ be expressed with a positive index. *Ans.* $\frac{1}{a^{\frac{1}{2}}}$

5. Let $\frac{1}{a+x}$ be expressed with a negative index.

Ans. $(a+x)^{-1}$.

6. Let $a(a^2-x^2)^{-\frac{1}{3}}$ be expressed with a positive index.

Ans. $\frac{a}{(a^2-x^2)^{\frac{1}{3}}}$.

CASE VIII.

To involve or raise Surd Quantities to any power.

RULE.

When the surd is a simple quantity, multiply its index by 2 for the square, by 3 for the cube &c., and it will give the power of the surd part, which, being annexed to the proper power of the rational prat, will give the whole power required. And if it be a compound quantity, mul tiply it by itself the proper number of times, according to the usual rule.*

EXAMPLES.

1. It is required to find the square of $\frac{2}{3}a^{\frac{1}{3}}$.

Here $\left(\frac{2}{3}a^{\frac{1}{3}}\right)^2=\frac{4}{9}a^{\frac{1}{3}\times 2}=\frac{4}{9}a^{\frac{2}{3}}=\frac{4}{9}\sqrt[3]{a^2}$. *Ans.*

2. It is required to find the cube of $\frac{2}{3}\sqrt{3}$.

* When any quantity that is affected with the sign of the square root, is to be raised to the second power, or squared, it is done by suppressing the sign. Thus, $(\sqrt{a})^2$, or $\sqrt{a}\times\sqrt{a}=a$; and $\sqrt{(a+b)}^2$ or $\sqrt{(a+b)}\times\sqrt{(a+b)}=a+b$.

Here $\frac{8}{27}\times 3^{\frac{3}{2}}=\frac{8}{27}\sqrt{27}=\frac{8}{27}\sqrt{(9\times 3)}=\frac{8}{9}\sqrt{3}$. *Ans.*

3. It is required to find the square of $3\sqrt[3]{3}$. *Ans.* $9\sqrt[3]{3^2}$.

4. It is required to find the cube of $17\sqrt{21}$.

Ans. $103173\times 21^{\frac{1}{2}}$.

5. It is required to find the 4th power of $\frac{1}{6}\sqrt{6}$.

Ans. $\frac{1}{36}$.

6. It is required to find the square of $3+2\sqrt{5}$.

Ans. $29+12\sqrt{5}$.

7. It is required to find the cube of $\sqrt{x}+3\sqrt{y}$.

Ans. $x\sqrt{x}+9x\sqrt{y}+27y\sqrt{x}+27y\sqrt{y}$.

8. It is required to find the 4th power of $\sqrt{3}-\sqrt{2}$.

Ans. $49-20\sqrt{6}$

CASE IX.

To find the Roots of Surd Quantities.

RULE.

When the surd is a simple quantity, multiply its index by $\frac{1}{2}$ for the square root, by $\frac{1}{3}$ for the cube root, &c., and it will give the root of the surd part; which being annexed to the root of the rational part, will give the whole root required. And if it be a compound quantity, find its root by the usual rule.*

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* The nth root of the mth power of any number a, or the mth power of the nth root of a, is $a^{\frac{m}{n}}$

Also, the nth root of the mth root of any number a, or the mth root of the nth root of a, is $a^{\frac{1}{mn}}$.

From the last expression, it appears that the square root of the square root of a is the 4th root of a; and that the cube root of the square root of a, or the square root of the cube root of a, is the 6th root of a; and so on for the fourth, fifth, or any other numerical root of this kind.

EXAMPLES.

1. It is required to find the square root of $9\sqrt[3]{3}$.

Here $(9\sqrt[3]{3})^{\frac{1}{2}}=9^{\frac{1}{2}}\times 3^{\frac{1}{3}\times\frac{1}{2}}=9^{\frac{1}{2}}\times 3^{\frac{1}{6}}=3\sqrt[6]{3}$. *Ans.*

2. It is required to find the cube root of $\frac{1}{8}\sqrt{2}$.

Here $\left(\frac{1}{8}\sqrt{2}\right)^{\frac{1}{3}}=\left(\frac{1}{8}\right)^{\frac{1}{3}}\times(2^{\frac{1}{2}\times\frac{1}{3}})=\frac{1}{2}(2^{\frac{1}{6}})=\frac{1}{2}\sqrt[6]{2}$. *Ans.*

3. It is required to find the square root of 10^3.

Ans. $10\sqrt{10}$.

4. It is required to find the cube root of $\frac{8}{27}a^4$.

Ans. $\frac{2}{3}a\sqrt[3]{a}$.

5. It is required to find the 4th root of $\frac{16}{81}a^{\frac{2}{3}}$.

Ans. $\frac{2}{3}\sqrt{a^{\frac{1}{3}}}$.

6. It is required to find the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$.

Ans. $\frac{1}{3}\sqrt{3a}$.

7. It is required to find the square root of $x^2-4x\sqrt{a}+4a$.

Ans. $x-2\sqrt{a}$.

8. It is required to find the square root of $a+2\sqrt{ab}+b$.

Ans. $\sqrt{a}+\sqrt{b}$

CASE X.

To transform a Binomial, or a residual Surd, into a general Surd.

RULE.

Involve the given binomial, or residual, to a powe

corresponding with that denoted by the surd; then set the radical sign of the same root over it, and it will be the general surd required.

EXAMPLES.

1. It is required to reduce $2+\sqrt{3}$ to a general surd.

Here $(2+\sqrt{3})^2=4+3+4\sqrt{3}=7+4\sqrt{3}$; therefore $2+\sqrt{3}=\sqrt{(7+4\sqrt{3})}$, the *Answer.*

2. It is required to reduce $\sqrt{2}+\sqrt{3}$ to a general surd.

Here $(\sqrt{2}+\sqrt{3})^2=2+3+2\sqrt{6}=5+2\sqrt{6}$; therefore $\sqrt{2}+\sqrt{3}=\sqrt{(5+2\sqrt{6})}$, the *Answer.*

3. It is required to reduce $\sqrt[3]{2}+\sqrt[3]{4}$ to a general surd.

Here $(\sqrt[3]{2}+\sqrt[3]{4})^3=6+6\sqrt[3]{2}+6\sqrt[3]{4}$; therefore $\sqrt[3]{2}+\sqrt[3]{4}=\sqrt[3]{\{6(1+\sqrt{2}+\sqrt{4})\}}$, the *Answer.*

4. It is required to reduce $3-\sqrt{5}$ to a general surd.

Ans. $\sqrt{(14-6\sqrt{5})}$.

5. It is required to reduce $\sqrt{2}-2\sqrt{6}$ to a general surd.

Ans. $\sqrt{(26-8\sqrt{3})}$.

6. It is required to reduce $4-\sqrt{7}$ to a general surd.

Ans. $\sqrt{(23-8\sqrt{7})}$.

7. It is required to reduce $2\sqrt[3]{3}-\sqrt[3]{9}$ to a general surd.

Ans. $\sqrt[3]{[15+18(2\sqrt[3]{3}-\sqrt[3]{9})]}$.

CASE XI.

To extract the Square Root of a binomial, or residual Surd.

RULE.*

Substitute the numbers, or parts, of which the given

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* Demon. Let $\sqrt{a+\sqrt{b}}=m+n$.

$\therefore \sqrt{a-\sqrt{b}}=m-n$.

And $\sqrt{a^2-b}=m^2-n^2$.

Also $2a=(m+n)^2+(m-n)^2=2m^2+2n^2$.

$\therefore a=m^2+n^2$.

$\therefore m=\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}$ and $n=\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}$.

$$\therefore \sqrt{a+\sqrt{b}}=m+n=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)}\}}+\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}$$

Similarly

$$\sqrt{a-\sqrt{b}}=m-n=\sqrt{\{\tfrac{1}{2}a+\sqrt{(a^2-b)}\}}-\sqrt{\{\tfrac{1}{2}a-\sqrt{(a^2-b)}\}}.$$

surd is composed, in the place of the letters, in one of the two following formulæ, according as it is a binomial or a residual, and it will give the root required.

$$\sqrt{(a+\sqrt{b})}=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)}\}}+\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}.$$
$$\sqrt{(a-\sqrt{b})}=\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}+\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}.$$

Where it is to be observed, that if both a and $\sqrt{(a^2-b)}$, in these formulæ, be rational quantities, the root will consist either of two surds, or of a rational part and a surd, which are the only cases of the rule that are useful.

EXAMPLES.

1. It is required to find the square root of $\sqrt{(11+6\sqrt{2})}$.

Here, $\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}$* $=\sqrt{\{\frac{11}{2}+\frac{1}{2}\sqrt{(121-72)}\}}=\sqrt{(\frac{11}{2}+\frac{7}{2})}=3$;

And, $\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{\frac{11}{2}-\frac{1}{2}\sqrt{(121-72)}\}}=\sqrt{(\frac{11}{2}-\frac{7}{2})}=\sqrt{2}$.

Whence $\sqrt{(11+6\sqrt{2})}=3+\sqrt{2}$, the answer required.

2. It is required to find the square root of $3-2\sqrt{2}$.

Here, $\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{\frac{3}{2}+\frac{1}{2}\sqrt{(9-8)}}=\sqrt{(\frac{3}{2}+\frac{1}{2})}=\sqrt{2}$; and $-\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}=-\sqrt{\{\frac{3}{2}-\frac{1}{2}\sqrt{(9-8)}\}}=-\sqrt{(\frac{3}{2}-\frac{1}{2})}=-1$;

Whence $3-2\sqrt{2}=\sqrt{2}-1$, the answer required.

3. It is required to find the square root of $6\pm2\sqrt{5}$.

Ans. $\sqrt{5}\pm1$.

4. It is required to find the square root of $23\pm8\sqrt{7}$.

Ans. $4\pm\sqrt{7}$.

5. It is required to find the square root of $36\pm10\sqrt{11}$.

Ans. $5\pm\sqrt{11}$.

6. It is required to find the square root of $33\pm12\sqrt{6}$.

Ans. $\sqrt{24}\pm3$.

CASE XII.

To find such a Multiplier, or Multipliers, as will make any binomial Surd rational.

RULE.

1. When one or both of the terms are any even roots,

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* It may here be observed, that b denotes the quantity under the second radical after its coefficient has been introduced. Thus, $b=72$, because $6\sqrt{2}=\sqrt{(36\times2)}=\sqrt{72}$.

multiply the given binomial, or residual, by the same expression, with the sign of one of its terms changed; and repeat the operation in the same way, as long as there are surds, when the last result will be rational.

2. When the terms of the binomial surd are odd roots, the rule becomes more complicated; but for the sum or difference of two cube roots, which is one of the most useful cases, the multiplier will be a trinomial surd, consisting of the squares of the two given terms and their product, with its sign changed.*

EXAMPLES.

1. To find a multiplier that shall render $5+\sqrt{3}$ rational.

Given surd $5+\sqrt{3}$
Multiplier $5-\sqrt{3}$

Product $25-3=22$, as required.

2. To find the multiplier that shall make $\sqrt{5}+\sqrt{3}$ rational.

Given surd $\sqrt{5}+\sqrt{3}$
Multiplier $\sqrt{5}-\sqrt{3}$

Product $5-3=2$, as required.

3. To find multipliers that shall make $\sqrt[4]{5}+\sqrt[4]{3}$ rational.

Given surd $\sqrt[4]{5}+\sqrt[4]{3}$
1st multiplier $\sqrt[4]{5}-\sqrt[4]{3}$

1st product $\sqrt{5}-\sqrt{3}$
2d multiplier $\sqrt{5}+\sqrt{3}$

2d product $5-3=2$, as required.

4. To find a multiplier that shall make $\sqrt[3]{7}+\sqrt[3]{3}$ rational.

* The following rule will be found more convenient than that given in the text, and answers equally for odd or even roots.

Rule.—Expand the given surd, with the sign of one of its terms changed, to a power one degree less than the denominator of the surd; the result, neglecting the coefficients, is the multiplier required.

$$\begin{array}{lllll}
\text{Given surd} & \sqrt[3]{7}+\sqrt[3]{3} & & & \\
\text{Multiplier} & \sqrt[3]{7^2}-\sqrt[3]{(7\times 3)}+\sqrt[3]{3^2} & & & \\
\hline
& 7+\sqrt[3]{(3\times 7^2)} & & & \\
& \quad -\sqrt[3]{(3\times 7^2)}-\sqrt[3]{(7\times 3^2)} & & & \\
& \qquad\qquad +\sqrt[3]{(7\times 3^2)}+3 & & & \\
\hline
\end{array}$$

Product $=7+3=10$, as was required.

or thus:

Given the surd $\sqrt[3]{7}+\sqrt[3]{3}$.
Square of the terms $\sqrt[3]{49}$ and $\sqrt[3]{9}$.
Product with sign changed $=-\sqrt[3]{21}$.
$\therefore \sqrt[3]{49}-\sqrt[3]{21}+\sqrt[3]{9}$ is the multiplier.

$$\begin{array}{ll}
\text{Given surd} & \sqrt[3]{7}+\sqrt[3]{3} \\
\hline
& 7-\sqrt[3]{147}+\sqrt[3]{63} \\
& \qquad \sqrt[3]{147}-\sqrt[3]{63}+3 \\
\hline
\text{Product} = & 7 \qquad\qquad\qquad +3=10, \text{ as before.}
\end{array}$$

5. To find a multiplier that shall make $\sqrt{5}-\sqrt{x}$ rational. *Ans.* $\sqrt{5}+\sqrt{x}$.

6. To find a multiplier that shall make $\sqrt{a}+\sqrt{b}$ rational *Ans.* $\sqrt{a}-\sqrt{b}$

7. To find a multiplier that shall make $a+\sqrt{b}$ rational. *Ans.* $a-\sqrt{b}$.

8. It is required to find a multiplier that shall make $1-\sqrt[3]{2a}$ rational. *Ans.* $1+\sqrt[3]{2a}+\sqrt[3]{4a^2}$.

9. It is required to find a multiplier that shall make $\sqrt[3]{3}-\frac{1}{2}\sqrt[3]{2}$ rational. *Ans.* $\sqrt[3]{9}+\frac{1}{2}\sqrt[3]{6}+\frac{1}{4}\sqrt[3]{4}$.

CASE XIII.

To reduce a Fraction, whose Denominator is either a simple or a compound Surd, to another that shall have a rational Denominator.

RULE.

1. When any simple fraction is of the form $\frac{b}{\sqrt{a}}$, multiply each of its terms by $\sqrt{a}$, and the resulting fraction will be $\frac{b\sqrt{a}}{a}$.

Or when it is of the form $\frac{b}{\sqrt[3]{a}}$, multiply them by $\sqrt[3]{a^2}$, and the result will be $\frac{b\sqrt[3]{a^2}}{a}$.

And for the general form, $\frac{b}{\sqrt[n]{a}}$, multiply by $\sqrt[n]{a^{n-1}}$, and the result will be $\frac{b\sqrt[n]{a^{n-1}}}{a}$.

2. If it be a compound surd, find such a multiplier, by the last rule, as will make the denominator rational; and multiply both the numerator and denominator by it, and the result will be the fraction required.

EXAMPLES.

1. Reduce the fractions $\frac{2}{\sqrt{3}}$ and $\frac{3}{\frac{1}{2}\sqrt[4]{5}}$ to others that shall have rational denominators.

Here $\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$; and $\frac{3}{\frac{1}{2}\sqrt[4]{5}}=\frac{3}{\frac{1}{2}\sqrt[4]{5}}\times\frac{\sqrt[4]{5^3}}{\sqrt[4]{5^3}}=$
$\frac{3\sqrt[4]{5^3}}{\frac{1}{2}\times 5}=\frac{6\sqrt[4]{5^3}}{5}=\frac{6}{5}\sqrt[4]{125}$, the answer required.

2. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to a fraction whose denominator shall be rational.

Here
$$\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=\frac{3\sqrt{5}+3\sqrt{2}}{3}=$$
$\sqrt{5}+\sqrt{2}$, the answer required.

3. Reduce $\frac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction whose denominator shall be rational.

Here
$$\frac{\sqrt{2}}{3-\sqrt{2}}=\frac{\sqrt{2}\times(3+\sqrt{2})}{(3-\sqrt{2})\times(3+\sqrt{2})}=\frac{3\sqrt{2}+2}{9-2}=\frac{2+3\sqrt{2}}{7}=$$
$\frac{2}{7}+\frac{3}{7}\sqrt{2}$, the answer required.

4. Reduce $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$ to a fraction that shall have a rational denominator. *Ans.* $\frac{\sqrt{42}-\sqrt{18}}{4}$.

5. Reduce $\frac{x}{3+\sqrt{x}}$ to a fraction that shall have a rational denominator. *Ans.* $\frac{3x-x\sqrt{x}}{9-x}$.

6. Reduce $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ to a fraction the denominator of which shall be rational. *Ans.* $\frac{a+b-2\sqrt{ab}}{a-b}$.

7. Reduce $\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}$ to a fraction that shall have a rational denominator. *Ans.* $5(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25})$.

8. Reduce $\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}$ to a fraction, that shall have a rational denominator. *Ans.* $\frac{\sqrt[3]{243}-\sqrt[3]{270}+\sqrt[3]{300}}{19}$.

9. Reduce $\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}$ to a fraction that shall have rational denominator. *Ans.* $4(\sqrt[4]{5}-\sqrt[4]{4})(2+\sqrt{5})$.

OF

ARITHMETICAL PROPORTION

AND PROGRESSION.

Arithmetical Proportion is the relation which two numbers, or quantities, of the same kind, have to two others, when the difference of the first pair is equal to that of the second.

Hence, three quantities are in arithmetical proportion when the difference of the first and second is equal to the difference of the second and third.

Thus, 2, 4, 6, and a, $a+b$, $a+2b$, are quantities in arithmetical proportion.

And four quantities are in arithmetical proportion when the difference of the first and second is equal to the difference of the third and fourth.

Thus, 3, 7, 12, 16, and a, $a+b$, c, $c+b$, are quantities in arithmetical proportion.

Arithmetical Progression is when a series of numbers, or quantities, increase or decrease by the same common difference.

Thus, 1, 3, 5, 7, 9, &c., and a, $a+d$, $a+2d$, $a+3d$, &c., are increasing series in arithmetical progression, the common differences of which are 2 and d.

And 15, 12, 9, 6, &c., and a, $a-d$, $a-2d$, $a-3d$, &c., are decreasing series in arithmetical progression, the common differences of which are 3 and d.

The most useful properties of arithmetical proportion and progression are contained in the following theorems:

1. If four quantities are in arithmetical proportion, the sum of the two extremes will be equal to the sum of the two means.

Thus, if the proportionals be 2, 5, 7, 10, or a, b, c, d; then will $2+10=5+7$, and $a+d=b+c$.

2. And if three quantities be in arithmetical proportion, the sum of the two extremes will be double the mean.

Thus, if the proportionals be 3, 6, 9, or a, b c, then will $3+9=2\times 6=12$, and $a+c=2b$.

3. Hence an arithmetical mean between any two quantities is equal to half the sum of those quantities.

Thus, an arithmetical mean between 2 and 4 is $=\frac{2+4}{2}=3$; and between 5 and 6 it is $=\frac{5+6}{2}=5\frac{1}{2}$.

And an arithmetical mean between a and b is $\frac{a+b}{2}$.

4. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two terms that are equally distant from them, or to double the middle term, when the number of terms is odd.

Thus, if the series be 2, 4, 6, 8, 10, then will $2+10=4+8=2\times 6=12$.

And, if the series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$, then will $a+(a+4d)=(a+d)+(a+3d)=2\times(a+2d)$.

5. The last term of any increasing arithmetical series is equal to the first term *plus* the product of the common difference by the number of terms less one; and if the series be decreasing, it will be equal to the first term *minus* that product.

Thus, the nth or last term (l) of the series a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c., is $a+(n-1)d$, or $l=a+(n-1)d$.

And the nth or last term (l) of the series a, $a-d$, $a-2d$, $a-3d$, $a-4d$, &c., is $a-(n-1)d$, or $l=a-(n-1)d$.

6. The sum of any series of quantities in arithmetical progression is equal to the sum of the two extremes multiplied by half the number of terms.

Thus, the sum of 2, 4, 6, 8, 10, 12, is $=(2+12)\times\frac{6}{2}=(2+12)\times 3=14\times 3=42$.

And if the series be $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$, &c.. . . . $+l$, and its sum be denoted by S, we shall have

$$S=(a+l)\times\frac{n}{2},$$

where l is the last term, and n the number of terms.

Or, the sum of any increasing arithmetical series may be found, without considering the last term, by adding the product of the common difference by the number of terms less one to twice the first term, and then multiplying the result by half the number of terms.

And, if the series be decreasing, its sum will be found by subtracting the above product from twice the first term, and then multiplying the result by half the number of terms, as before.

Thus, if the series be $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$, &c., continued to n terms, we shall have

$$S=\{2a+(n-1)d\}\times\frac{n}{2}.$$

And if the series be $a+(a-d)+(a-2d)+(a-3d)+(a-4d)$, &c., to n terms, we shall have

$$S=\{2a-(n-1)d\}\times\frac{n}{2}.*$$

* The sum of any number of terms (n) of the series of natural numbers 1, 2, 3, 4, 5, 6, 7, &c., is $=\frac{n(n+1)}{2}$.

EXAMPLES.

1. The first term of an increasing arithmetical series is 3, the common difference 2, and the number of terms 20; required the sum of the series.

$$S=[2a+(n-1)d]\times\frac{n}{2}=[2\times3+(20-1)\times2]\times\frac{20}{2}=$$

$[6+(19\times2)]\times10=(6+38)\times10=44\times10=440$, the sum required.

2. The first term of a decreasing arithmetical series is 100, the common difference 3, and the number of terms 34; required the sum of the series.

$$S=[2a-(n-1)d]\times\frac{n}{2}=[(2\times100)-(34-1)\times3]\times\frac{34}{2}=$$

$[200-(33\times3)]\times17=(200-99)\times17=101\times17=1717$, the sum required.

3. Required the sum of the natural numbers 1, 2, 3, 4, 5, 6, &c., continued to 1000 terms. *Ans.* 500500.

4. Required the sum of the odd numbers 1, 3, 5, 7, 9, &c, continued to 101 terms. *Ans.* 10201.

5. How many strokes does the clock in Venice strike in a day, which goes on to 24 o'clock? *Ans.* 300.

6. Required the 365th term of the series of even numbers 2, 4, 6, 8, 10, 12, &c. *Ans.* 730.

Thus $1+2+3+4+5$, &c., continued to 100 terms, is $=\frac{100\times101}{2}$ $=50\times101=5050$.

Also the sum of any number of terms (n) of the series of odd numbers 1, 3, 5, 7, 9, 11, &c., is $=n^2$.

Thus, $1+3+5+7+9$, &c., continued to 50 terms, is $=50^2=2500$.

And if any three of the quantities a, d, n, S, be given, the fourth may be found from the equation

$$S=\left\{2a\pm(n-1)d\right\}\times\frac{n}{2},\text{ or }(a+l)\times\frac{n}{2},$$

where the upper sign + is to be used when the series is increasing, and the lower sign − when it is decreasing; also the last term $l=a\pm(n-1)d$, as above.

7. The first term of a decreasing arithmetical series is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the sum of the series. *Ans.* 140.

8. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other; how far will a person travel, who shall bring them one by one to a basket, placed at a distance of a yard from the first stone? *Ans.* 5 miles and 1300 yards.

OF

GEOMETRICAL PROPORTION

AND

PROGRESSION.

Geometrical Proportion is the relation which two numbers, or quantities, of the same kind have to two others, when the antecedents, or leading terms of each pair, are the same parts of their consequents, or the consequents of the antecedents.

And if two quantities only are to be compared together, the part, or parts, which the antecedent is of its consequent, or the consequent of the antecedent, is called the ratio; observing, in both cases, always to follow the same method.

Hence, three quantities are in geometrical proportion when the first is the same part, or multiple, of the second, as the second is of the third.

Thus, 3, 6, 12, and a, ar, ar^2, are quantities in geometrical proportion.

And four quantities are in geometrical proportion when the first is the same part, or multiple, of the second, as the third is of the fourth.

Thus 2, 8, 3, 12, and a, ar, b, br, are geometrical proportionals.

Direct proportion is when the same relation subsists between the first of four terms and the second, as between the third and fourth.

Thus, 3, 6, 5, 10, and a, ar, b, br, are in direct proportion.

Inverse, or reciprocal proportion, is when the first and second of four quantities are directly proportional to the reciprocals of the third and fourth.

Thus, 2, 6, 9, 3, and a, ar, br, b, are inversely proportional; because 2, 6, $\frac{1}{9}$, $\frac{1}{3}$, and a, ar, $\frac{1}{br}$, $\frac{1}{b}$, are directly proportional.

GEOMETRICAL PROGRESSION is when a series of numbers, or quantities, have the same constant ratio; or which increase, or decrease, by a common multiplier, or divisor.

Thus, 2, 4, 8, 16, 32, 64, &c., and a, ar, ar^2, ar^3, ar^4, &c., are series in geometrical progression.

The most useful properties of geometrical proportion and progression are contained in the following theorems:

1. If three quantities be in geometrical proportion, the product of the two extremes will be equal to the square of the mean.

Thus, if the proportionals be 2, 4, 8, or a, b, c, then will $2\times 8=4^2$ and $a\times c=b^2$.

2. Hence, a geometrical mean proportional, between any two quantities, is equal to the square root of their product.

Thus, a geometric mean between 4 and 9 is $=\sqrt{36}=6$; and between $\frac{1}{2}$ and $\frac{1}{8}$, it is $=\sqrt{\frac{1}{16}}$, or $\frac{1}{4}$.

Also, a geometric mean between a and b is $=\sqrt{ab}$.

3. If four quantities be in geometrical proportion, the product of the two extremes will be equal to that of the means.

Thus, if the proportionals be 2, 4, 6, 12, or a, b, c, d; then will $2\times 12=4\times 6$, and $a\times d=b\times c$.

4. Hence the product of the means of four proportional quantities divided by either of the extremes, will give the other extreme; and the product of the extremes, divided by either of the means, will give the other mean.

Thus, if the proportionals be 3, 9, 5, 15, or a, b, c, d; then will $\frac{9\times 5}{3}=15$, and $\frac{3\times 15}{5}=9$

And $\frac{b\times c}{a}=d$, and $\frac{a\times d}{c}=b$.

5. Also, if any two products be equal to each other, either of the terms of one of them will be to either of the terms of the other, as the remaining term of the latter is to the remaining term of the first.

Thus, if $ad=bc$, or $2\times 15=6\times 5$, then will any of the following forms of these quantities be proportional:

Directly, $a:b::c:d$, or $2:6::5:15$.

Invertedly, $b:a::d:c$, or $6:2::15:5$.

Alternately, $a:c::b:d$, or $2:5::6:15$.

Conjunctly, $a:a+b::c:c+d$, or $2:8::5:20$.

Disjunctly, $a:b\backsim a::c:d\backsim c$, or $2:4::5:10$.

Mixedly, $b+a:b\backsim a::d+c:d\backsim c$, or $8:4::20:10$.

In all of which cases, the product of the two extremes is equal to that of the two means.

6. In any continued geometrical series, the product of the two extremes is equal to the product of any two means that are equally distant from them; or to the square of the mean, when the number of terms is odd.

Thus, if the series be 2, 4, 8, 16, 32; then will

$$2\times 32=4\times 16=8^2.$$

7. In any geometrical series, the last term is equal to the product arising from multiplying the first term by such a power of the ratio as is denoted by the number of terms less one.

Thus, in the series 2, 6, 18, 54, 162, where 3 is the ratio, we shall have $2\times 3^4=2\times 81=162$.

And in the series a, ar, ar^2, ar^3, ar^4, &c., continued to n terms, where r is the ratio, the nth, or last term (l), will be

$$l=ar^{n-1}.$$

8. The sum of any series of quantities in geometrical progression, either increasing or decreasing, is found by multiplying the last term by the ratio, and then dividing the difference of this product and the first term, by the difference between the ratio and unity.

Thus, in the series 2, 4, 8, 16, 32, 64, 128, 256, 512, where 2 is the ratio, we shall have $\dfrac{512\times 2-2}{2-1}=1024-2$ $=1022$, the sum of the terms.

Or, a similar rule, without considering the last term, may be expressed thus

Find such a power of the ratio as is denoted by the number of terms of the series; then divide the difference between this power and unity, by the difference between the ratio and unity, and the result, multiplied by the first term, will be the sum of the series.

Thus, in the series $a+ar+ar^2+ar^3+ar^4 \ldots +ar^{n-1}$, continued to n terms, we shall have

$$S=a\left(\frac{r^n-1}{r-1}\right).$$

And if the ratio, or common multiplier, r, in this last series, be a proper fraction, and consequently the series a decreasing one, we shall have, in that case,

$$a+ar+ar^2+ar^3+ar^4+\&c., \textit{ ad infinitum} = \frac{a}{1-r}.$$

EXAMPLES.

1. The first term of a geometrical series is 1, the ratio 2, and the number of terms 10; what is the sum of the series?

$S=\frac{a(r^n-1)}{r-1}=\frac{1\times(2^{10}-1)}{2-1}=\frac{1024-1}{2-1}=1023$, the sum required.

2. The first term of a geometrical series is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5; required the sum of the series.

$S=\frac{a(1-r^n)}{1-r}=\frac{\frac{1}{2}[1-(\frac{1}{3})^5]}{1-\frac{1}{3}}=\frac{\frac{1}{2}(1-\frac{1}{243})}{\frac{2}{3}}=\frac{242}{243}\times\frac{3}{4}=\frac{121}{162}$, the sum.

3. Required the sum of 1, 2, 4, 8, 16, 32, &c., continued to 20 terms. *Ans.* 1048575.

4. Required the sum of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c., continued to 8 terms. *Ans.* $1\frac{127}{128}$

5. Required the sum of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$ $\frac{1}{81}$, &c., continued to 10 terms. *Ans.* $1\frac{6561}{17122}$.

6. A person being asked to dispose of a fine horse, said he would sell him on condition of having a farthing for the first nail in his shoes, a halfpenny for the second, a penny for the third, twopence for the fourth, and so on, doubling the price of every nail, to 32, the number of nails in his four shoes; what would the horse be sold for at that rate? *Ans.* 4473924*l.* 5*s.* $3\frac{3}{4}$*d.*

OF EQUATIONS.

The Doctrine of Equations is that branch of algebra which treats of the methods of determining the values of unknown quantities by means of their relations to others which are known.

This is done by making certain algebraic expressions equal to each other, (which formula, in that case, is called an equation,) and then working by the rules of the art, till the quantity sought is found equal to some given quantity, and consequently becomes known.

The terms of an equation are the quantities of which it is composed; and the parts that stand on the right and left of the sign $=$ are called the two members, or sides, of the equation.

Thus, if $x=a+b$, the terms are x, a, and b; and the meaning of the expression is, that some quantity, x, standing on the left-hand side of the equation, is equal to the sum of the quantities a and b on the right-hand side.

A simple equation is that which contains only the first power of the unknown quantity; as,

$$x+a=3b, \text{ or } ax=bc, \text{ or } 2x+3a^2=5b^2;$$

where x denotes the unknown quantity, and the other letters, or numbers, the known quantities.

A compound equation is that which contains two or more different powers of the unknown quantity; as,

$$x^2+ax=b, \text{ or } x^3-4x^2+3x=25.$$

Equations are also divided into different orders, or receive particular names, according to the highest power of the unknown quantity contained in any one of their terms: as, quadratic equations, cubic equations, biquadratic equations, &c.

Thus, a quadratic equation is that in which the unknown quantity is of two dimensions, or which rises to the second power: as,

$$x^3=20\ ;\ x^2+ax=b,\text{ or }3x^2+10x=100.$$

A cubic equation is that in which the unknown quantity is of three dimensions, or which rises to the third power: as,

$$x^3=27\ ;\ 2x^2-3x=35\ ;\text{ or }x^3-ax^2+bx=c$$

A biquadratic equation is that in which the unknown quantity is of four dimensions, or which rises to the fourth power: as,

$$x^4=25\ ;\ 5x^4-4x=6\ ;\text{ or }x^4-ax^3+bx^2-cx=d.$$

And so on for equations of the 5th, 6th, and other higher orders, which are all denominated according to the highest power of the unknown quantity contained in any one of their terms.

The root of an equation is such a number or quantity as, being substituted for the unknown quantity, will make both sides of the equation vanish, or become equal to each other.

A simple equation can have only one root; but every compound equation has as many roots as it contains dimensions, or as is denoted by the index of the highest power of the unknown quantity, in that equation.

Thus, in the quadratic equation $x^2+2x=15$, the root, or value of x, is either $+3$ or -5 and in the cubic equation $x^3-9x^2+26x=24$, the roots are 2, 3, and 4, as will be found by substituting each of these numbers for x.

In an equation of an odd number of dimensions, one of its roots will always be real; whereas, in an equation of an even number of dimensions, all its roots may be imaginary; as roots of this kind always enter into an equation by pairs.

Such are the equations $x^2-6x+14=0$, and $x^4-2x^3-9x^2+10x+50=0$.*

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* To the properties of equations above mentioned, we may here further add,

1. That the sum of all the roots of any equation is equal to the coefficient of the second term of that equation, with its sign changed.

2. The sum of the products of every two of the roots is equal to the coefficient of the third term, without any change in the sign.

OF THE

RESOLUTION OF SIMPLE EQUATIONS,

Containing only one unknown Quantity.

THE solution of simple, as well as of other equations, is the disengaging the unknown quantity, in all such expressions, from the other quantities with which it is connected, and making it stand alone on one side of the equation, so as to be equal to such as are known on the other side; for the performing of which, several axioms and processes are required, the most useful and necessary of which are the following.*

CASE I.

Any quantity may be transposed from one side of an equation to the other, by changing its sign; and the two members, or sides, will still be equal.

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3. The sum of the products of every three of the roots is equal to the coefficient of the fourth term, with its sign changed.

4. And so on, to the last, or absolute term, which is equal to the product of all the roots, with the sign changed or not, according as the equation is of an odd or an even number of dimensions.

See, for a more particular account of the general theory of equations, Vol. II. of my *Treatise on Algebra,* 8vo. 2d Ed. 1820.

* The operations required, for the purpose here mentioned, are chiefly such as are derived from the following simple and evident principles:—

1. If the same quantity be added to, or subtracted from, each of two equal quantities, the results will still be equal; which is the same, in effect, as taking any quantity from one side of an equation, and placing it on the other side, with a contrary sign.

2. If all the terms of any two equal quantities be multiplied or divided by the same quantity, the products or quotients thence arising will be equal.

3. If two quantities, either simple or compound, be equal to each other, any like powers, or roots of them, will also be equal.

Each of these axioms, and others of a similar kind, will be found sufficiently illustrated by the processes arising out of the several examples annexed to the six different cases given in the text; as well as in other parts of the present performance.

Thus, if $x+3=7$; then will $x=7-3$, or $x=4$.

And, if $x-4+6=8$; then will $x=8+4-6=6$.

Also, if $x-a+b=c-d$; then will $x=a-b+c-d$.

And, if $4x-8=3x+20$; then $4x-3x=20+8$, and consequently $x=28$.

From this rule it also follows, that if a quantity be found on each side of an equation, with the same sign, it may be left out of both of them; and that the signs of all the terms of any equation may be changed from $+$ to $-$, or from $-$ to $+$, without altering its value.

Thus, if $x+5=7+5$; then, by cancelling, $x=7$.

And, if $a-x=b-c$; then, by changing the signs, $x-a=c-b$, or $x=a+c-b$.

CASE II.

If the unknown quantity, in any equation, be multiplied by any number, or quantity, the multiplier may be taken away, by dividing all the rest of the terms by it; and if it be divided by any number, the divisor may be taken away, by multiplying all the other terms by it.

Thus, if $ax=3ab-c$; then will $x=3b-\frac{c}{a}$.

And if $2x+4=16$; then will $x+2=8$,
or $x=8-2=6$.

Also, if $\frac{x}{2}=5+3$; then will $x=10+6=16$.

And, if $\frac{2x}{3}-2=4$; then $2x-6=12$, or, by division, $x-3=6$, or $x=9$.

CASE III.

Any equation may be cleared of fractions, by multiplying each of its terms, successively, by the denominators of those fractions; or by multiplying both sides by the product of all the denominators, or by any quantity that is a multiple of them.

Thus, if $\frac{x}{3}+\frac{x}{4}=5$, then, multiplying by 3, we have $x+\frac{3x}{4}=15$; and this, multiplied by 4, gives $4x+3x=60$; whence by addition, $7x=60$, or $x=\frac{60}{7}=8\frac{4}{7}$.

And, if $\frac{x}{4}+\frac{x}{6}=10$; then, multiplying by 12 (which is a multiple of 4 and 6,) $3x+2x=120$, or $5x=120$, or $x=\frac{120}{5}=24$.

It also appears, from this rule, that if the same number or quantity be found in each of the terms of an equation, either as a multiplier or divisor, it may be expunged from all of them, without altering the result.

Thus, if $ax=ab+ac$; then, by cancelling, $x=b+c$.

And, if $\frac{x}{a}+\frac{b}{a}=\frac{c}{a}$; then, $x+b=c$, or $x=c-b$.

CASE IV.

If the unknown qnantity, in any equation, be in the form of a surd, transpose the terms so that this may stand alone, on one side of the equation, and the remaining terms on the other (by Case I.); then involve each of the sides to such a power as corresponds with the index of the surd, and the equation will be rendered free from any irrational expression.

Thus, if $\sqrt{x}-2=3$; then will $\sqrt{x}=3+2=5$, or by squaring, $x=5^2=25$.

And, if $\sqrt{(3x+4)}=5$; then will $3x+4=25$, or $3x=25-4=21$, and consequently $x=\frac{21}{3}=7$.

Also, if $\sqrt[3]{(2x+3)}+4=8$; then will $\sqrt[3]{(2x+3)}=8-4=4$, or $2x+3=4^3=64$, and consequently $2x=64-3=61$, or $x=\frac{61}{2}=30\frac{1}{2}$.

CASE V.

If that side of the equation, which contains the unknown quantity, be a complete power, the equation may be reduced to a lower dimension, by extracting the root of the said power on both sides of the equation.

Thus, if $x^2=81$, then $x=\sqrt{81}=9$; and if $x^3=27$, then $x=\sqrt[3]{27}=3$.

Also, if $3x^2-9=24$; then $3x^2=24+9=33$, or $x^2=\frac{33}{3}=11$, and consequently $x=\sqrt{11}$.

And, if $x^2+6x+9=27$; then, since the left-hand side of the equation is a complete square, we shall have, by extracting the roots, $x+3=\sqrt{27}=\sqrt{(9\times3)}=3\sqrt{3}$, or $x=3\sqrt{3}-3$.

CASE VI.

Any analogy or proportion may be converted into an equation, by making the product of the two extreme terms equal to that of the two means.

Thus, if $3x : 16 :: 5 : 6$; then $3x\times6=16\times5$, or $18x=80$, or $x=\frac{80}{18}=\frac{40}{9}=4\frac{4}{9}$.

And, if $\frac{2x}{3} : a :: b : c$; then will $\frac{2cx}{3}=ab$, or $2cx=3ab$; or, by division, $x=\frac{3ab}{2c}$.

Also, if $12-x : \frac{x}{2} :: 4 : 1$; then $12-x=\frac{4x}{2}=2x$, or $2x+x=12$; and consequently $x=\frac{12}{3}=4$.

MISCELLANEOUS EXAMPLES.

1. Given $5x-15=2x+6$, to find the value of x.

Here $5x-2x=6+15$, or $3x=6+15=21$; and therefore, by division, we shall have $x=\frac{21}{3}=7$. *Ans.*

2. Given $40-6x-16=120-14x$, to find the value of x.

Here $14x-6x=120-40+16$; or $8x=136-40=96$; and therefore, by division, $x=\frac{96}{8}=12$. *Ans.*

3. Given $3x^2-10x=8x+x^2$, to find the value of x.

Here $3x-10=8+x$, by dividing by x; or $3x-x=8+10=18$, by transposition.

And consequently $2x=18$, or by division, $x=\frac{18}{2}=9$. *Ans.*

4. Given $6ax^3-12abx^2=3ax^3+6ax^2$, to find the value of x.

Here $2x-4b=x+2$, by dividing by $3ax^2$; or $2x-x=2+4b$; and therefore $x=4b+2$. *Ans.*

5. Given $x^2+2x+1=16$, to find the value of x.

Here $x+1=4$, by extracting the square root of each side.

And therefore, by transposition, $x=4-1=3$. *Ans.*

6. Given $5ax-3b=2dx+c$, to find the value of x.

Here $5ax-2dx=c+3b$; or $(5a-2d)x=c+3b$; and therefore, by division, $x=\frac{c+3b}{5a-2d}$. *Ans.*

7. Given $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10$, to find the value of x.

Here $x-\frac{2x}{3}+\frac{2x}{4}=20$; and $3x-2x+\frac{6x}{4}=60$; or $12x-8x+6x=240$; whence $10x=240$, or $x=24$. *Ans.*

8. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$, to find the value of x.

Here $x-3+\frac{2x}{3}=40-x+19$; or $3x-9+2x=120-3x+57$; whence $3x+2x+3x=120+57+9$; that is, $8x=186$, or $x=23\frac{1}{4}$. *Ans.*

9. Given $\sqrt{\frac{2x}{3}}+5=7$, to find the value of x.

Here $\sqrt{\frac{2x}{3}}=7-5=2$; whence, by squaring, $\frac{2x}{3}=2^2=4$, and $2x=12$, or $x=6$. *Ans.*

10. Given $x+\sqrt{(a^2+x^2)}=\frac{2a^2}{\sqrt{(a^2+x^2)}}$, to find the value of x.

Here $x\sqrt{(a^2+x^2)}+a^2+x^2=2a^2$; or $x\sqrt{(a^2+x^2)}=a^2-x^2$ and $x^2(a^2+x^2)=a^4-2a^2x^2+x^4$; whence $a^2x^2+x^4=a^4-2a^2x^2+x^4$, and $a^2x^2=a^4-2a^2x^2$

Therefore $3a^2x^2=a^4$, or $x^2=\frac{a^4}{3a^2}=\frac{a^2}{3}$; and consequently $x=\sqrt{\frac{a^2}{3}}=\sqrt{\frac{3a^2}{9}}=\frac{a}{3}\sqrt{3}$, *the Answer.*

EXAMPLES FOR PRACTICE.

1. Given $5x+22-2x=31$, to find the value of x.

Ans. $x=3$

2. Given $4-9y=14-11y$, to find the value of y.

Ans. $y=5$.

3. Given $x+18=3x-5$, to find the value of x.

Ans. $x=11\frac{1}{2}$.

4. Given $x+\frac{x}{2}+\frac{x}{3}=11$, to determine the value of x

Ans. $x=6$.

5. Given $2x-\frac{x}{2}+1=5x-2$, to find the value of x.

Ans. $x=\frac{6}{7}$.

6. Given $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{7}{10}$, to determine the value of x

Ans. $x=1\frac{1}{5}$.

7. Given $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$, to find the value of x.

Ans. $x=3\frac{6}{13}$.

8. Given $2+\sqrt{3x}=\sqrt{(4+5x)}$, to find the value of x.
Ans. $x=12$.

9. Given $x+a=\dfrac{x^2}{a+x}$, to find the value of x.
Ans. $x=-\dfrac{a}{2}$.

10. Given $\sqrt{x}+\sqrt{(a+x)}=\dfrac{2a}{\sqrt{(a+x)}}$, to find the value of x.
Ans. $x=\dfrac{a}{3}$.

11. Given $\dfrac{ax-b}{4}+\dfrac{a}{3}=\dfrac{bx}{2}-\dfrac{bx-a}{3}$, to find the value of x.
Ans. $x=\dfrac{3b}{3a-2b}$.

12. Given $\sqrt{(a^2+x^2)}=\sqrt[4]{(b^4+x^4)}$, to find the value of x.
Ans. $x=\sqrt{\dfrac{b^4-a^4}{2a^2}}$.

13. Given $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt{ax}$, to find the value of x.
Ans. $x=\dfrac{4a^2}{a^2+4}$.

14. Given $\dfrac{a}{1+x}+\dfrac{a}{1-x}=b$, to determine the value of x.
Ans. $x=\sqrt{\dfrac{b-2a}{b}}$.

15. Given $a+x=\sqrt{\{a^2+x\sqrt{(b^2+x^2)}\}}$, to find the value of x.
Ans. $x=\dfrac{b^2}{4a}-a$.

16. Given $\frac{1}{2}\sqrt{(x^2+3a^2)}+\frac{1}{2}\sqrt{(x^2-3a^2)}=x\sqrt{a}$, to find the value of x.
Ans. $x=\sqrt[4]{\dfrac{9a^3}{4-4a}}$.

17. Given $\sqrt{(a+x)}+\sqrt{(a-x)}=b$, to find the value of x.
Ans. $x=\dfrac{b}{2}\sqrt{(4a-b^2)}$.

18. Given $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$, to find the value of x.
Ans. $x=\sqrt{\left\{a^2-\left(\dfrac{b^3-2a}{3b}\right)^3\right\}}$.

19. Given $\sqrt{a}+\sqrt{x}=\sqrt{ax}$, to find the value of x.

$$Ans.\ x=\frac{a}{(\sqrt{a}-1)^2}.$$

20. Given $\sqrt{\frac{x+1}{x-1}}+\sqrt{\frac{x-1}{x+1}}=a$, to determine the value of x.

$$Ans.\ x=\frac{a}{\sqrt{a^2-4}}.$$

21. Given $\sqrt{(a^2+ax)}=a-\sqrt{(a^2-ax)}$, to find the value of x.

$$Ans.\ x=\frac{a}{2}\sqrt{3}.$$

22. Given $\sqrt{(a^2-x^2)}+x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}$, to find the value of x.

$$Ans.\ x=\left(\frac{a^2-1}{a^2+3}\right)^{\frac{1}{2}}.$$

23. Given $\sqrt{(x+a)}=c-\sqrt{(x+b)}$, to find the value of x.

$$Ans.\ x=\left(\frac{c^2+b-a}{2c}\right)^2-b.$$

24. Given $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}$, to find the value of x.

$$Ans.\ x=a\left(\frac{b+c}{b-c}\right).$$

Of the resolution of Simple Equations, containing two unknown quantities.

When there are two unknown quantities, and two independent simple equations involving them, they may be reduced to one, by any of the three following rules.

RULE I.

Observe which of the unknown quantities is the least involved, and find its value in each of the equations, as if the other was known, by the methods already explained; then let the two values, thus found, be put equal to each other, and there will arise a new equation with only one

unknown quantity in it, the value of which may be found as before.*

EXAMPLES.

1. Given $\begin{cases} 2x+3y=23 \\ 5x-2y=10 \end{cases}$ to find the values of x and y.

Here, from the first equation, $x=\frac{23-3y}{2}$,

and from the second equation, $x=\frac{10+2y}{5}$.

Whence by equality we have $\frac{23-3y}{2}=\frac{10+2y}{5}$,

or $115-15y=20+4y$, or $19y=115-20=95$,

$$\therefore \quad y=\frac{95}{19}=5, \text{ and } x=\frac{23-15}{2}=4.$$

2. Given $\begin{cases} x+y=a \\ x-y=b \end{cases}$ to find the values of x and y.

Here, from the first equation, $x=a-y$,

and from the second, $x=b+y$.

Whence $a-y=b+y$, or $2y=a-b$.

$$\text{Therefore } y=\frac{a-b}{2},$$

$$\text{and } x=a-y=a-\frac{a-b}{2}=\frac{a+b}{2}.$$

3. Given $\begin{cases} \frac{1}{2}x+\frac{1}{3}y=7 \\ \frac{1}{3}x+\frac{1}{2}y=8 \end{cases}$ to find the values of x and y.

Here, from the first equation, $x=14-\frac{2y}{3}$,

and from the second, $x=24-\frac{3y}{2}$.

Therefore, by equality, $14-\frac{2y}{3}=24-\frac{3y}{2}$,

* This rule depends upon the well-known axiom, that things which are equal to the same thing, are equal to each other; and the two following methods are founded on principles which are equally simple and obvious.

and consequently $42-2y=72-\frac{9y}{2}$,

and $84-4y=144-9y$.

Therefore, $5y=144-84=60$,

or, by division, $y=\frac{60}{5}=12$, and $x=14-\frac{24}{3}=6$

RULE II.

Find the value of either of the unknown quantities in that equation in which it is the least involved, as in the last rule; then substitute this value in the place of its equal in the other equation, and there will arise a new equation with only one unknown quantity in it, the value of which may be found as before.

EXAMPLES.

1. Given $\left\{\begin{array}{r} x+2y=17 \\ 3x-\ y=\ 2 \end{array}\right\}$ to find the values of x and y.

Here from the first equation, $x=17-2y$; which value being substituted for x in the second,

gives $3(17-2y)-y=2$,

or $51-6y-y=2$, or $7y=51-2=49$.

Whence $y=\frac{49}{7}=7$, and $x=17-2y=3$.

2. Given $\left\{\begin{array}{r} x+y=13 \\ x-y=\ 3 \end{array}\right\}$ to find the values of x and y.

Here from the first equation, $x=13-y$; which value, being substituted for x in the second,

gives $13-y-y=3$, or $2y=13-3=10$.

Whence $y=\frac{10}{2}=5$, and $x=13-y=8$.

3. Given $\left\{\begin{array}{c} x : y :: a : b \\ x^2+y^2=c \end{array}\right\}$ to find the values of x and y

Here the analogy turned into an equation,

gives $bx=ay$, or $x=\frac{ay}{b}$,

and this value, substituted for x in the second,

gives $\left(\frac{ay}{b}\right)^2+y^2=c$, or $\frac{a^2y^2}{b^2}+y^2=c$.

Whence we have $a^2y^2+b^2y^2=b^2c$, or $y^2=\frac{b^2c}{a^2+b^2}$,

and consequently, $y=b\sqrt{\frac{c}{a^2+b^2}}$, and $x=a\sqrt{\frac{c}{a^2+b^2}}$.

RULE III.

Let one or both of the given equations be multiplied, or divided, by such numbers, or quantities, as will make the term that contains one of the unknown quantities the same in each of them; then, by adding, or subtracting, the two equations thus obtained, as the case may require, there will arise a new equation, with only one unknown quantity in it, which may be resolved as before.

EXAMPLES.

1. Given $\left\{\begin{array}{r}3x+5y=40\\ x+2y=14\end{array}\right\}$ to find the values of x and y.

First, multiply the second equation by 3, and it will give $3x+6y=42$.

Then, subtract the first equation from this, and it will give $6y-5y=42-40$, or $y=2$.

Whence, also, $x=14-2y=14-4=10$.

2. Given $\left\{\begin{array}{r}5x-3y=9\\ 2x+5y=16\end{array}\right\}$ to find the values of x and y.

Multiply the first equation by 2, and the second by 5; then $10x-6y=18$, and $10x+25y=80$.

And if the former of these be subtracted from the latter, there will arise $31y=62$, or $y=\frac{62}{31}=2$.

Whence, by the first equation, $x=\frac{9+3y}{5}=\frac{15}{5}=3$.

EXAMPLES FOR PRACTICE.

1. Given $4x+y=34$, and $4y+x=16$, to find the values of x and y. *Ans.* $x=8$, $y=2$.

2. Given $2x+3y=16$, and $3x-2y=11$, to find the values of x and y. *Ans.* $x=5$, $y=2$.

3. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$, to find the values of x and y. *Ans.* $x=\frac{1}{2}$, $y=\frac{1}{3}$.

4. Given $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$, to find the values of x and y. *Ans.* $x=7$, $y=14$.

5. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, to find the values of x and y. *Ans.* $x=60$, $y=40$.

6. Given $x+y=s$, and $x^2-y^2=d$, to find the values of x and y. *Ans.* $x=\frac{s^2+d}{2s}$, $y=\frac{s^2-d}{2s}$.

7. Given $x+y:a::x-y:b$, and $x^2-y^2=c$, to find the values of x and y.

Ans. $x=\frac{a+b}{2}\sqrt{\frac{c}{ab}}$, $y=\frac{a-b}{2}\sqrt{\frac{c}{ab}}$.

8. Given $ax+by=c$, and $dx+ey=f$, to find the values of x and y. *Ans.* $x=\frac{ce-bf}{ae-bd}$, $y=\frac{af-dc}{ae-bd}$.

9. Given $x^2+y^2=a$, and $x^2-y^2=b$, to find the values of x and y. *Ans.* $x=\sqrt{\frac{a+b}{2}}$, $y=\sqrt{\frac{a-b}{2}}$.

10. Given $x^2+xy=a$, and $y^2+xy=b$, to find the values of x and y. *Ans.* $x=\frac{a}{\sqrt{(a+b)}}$, $y=\frac{b}{\sqrt{(a+b)}}$.

Of the resolution of Simple Equations, containing three or more unknown quantities.

When there are three unknown quantities, and three independent simple equations containing them, they may be reduced to one, by the following method.

RULE.

Find the values of one of the unknown quantities, in each of the three given equations, as if all the rest were known; then put the first of these values equal to the second, and either the first or second equal to the third, and there will arise two new equations with only two unknown quantities in them, the values of which may be found as in the former case; and thence the value of the third.

Or, multiply each of the equations by such numbers, or quantities, as will make one of their terms the same in each; then, having subtracted any two of these resulting equations from the third, or added them together, as the case may require, there will remain only two equations, which may be resolved by the former rules.

And in nearly the same way may four, five, &c., unknown quantities be exterminated from the same number of independent simple equations; but, in cases of this kind, there are frequently shorter and more commodious methods of operation, which can only be learnt from practice.

EXAMPLES.

1. Given $\left\{\begin{array}{r} x+\ y+\ z=29 \\ x+2y+3z=62 \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10 \end{array}\right\}$ to find x, y, and z.

Here, from the first equation, $x=29-y-z$,

from the second, $x=62-2y-3z$,

and from the third, $x=20-\frac{2}{3}y-\frac{1}{2}z$.

Whence $29-y-z=62-2y-3z$,

and $29-y-z=20-\frac{2}{3}y-\frac{1}{2}z$.

From the first of which $y=33-2z$,

and from the second, $y=27-\frac{3}{2}z$

Therefore $33-2z=27-\frac{3}{2}z$, or $z=12$

whence $y=33-2z=9$,

and $x=29-y-z=8$

2. Given $\begin{cases} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-\ \ z=63 \end{cases}$ to find x, y, and z.

Here, multiplying the first equation by 6, the second by 3, and the third by 2, we shall have

$$\begin{aligned} 12x+24y-18z&=132, \\ 12x-\ \ 6y+15z&=\ \ 54, \\ 12x+14y-\ \ 2z&=126. \end{aligned}$$

And, subtracting the second of these equations successively from the first and third, there will arise

$$\begin{aligned} 30y-33z&=78, \\ 20y-17z&=72. \end{aligned}$$

Or, by dividing the first of these two equations by 3, and then multiplying the result by 2,

$$\begin{aligned} 20y-22z&=52, \\ 20y-17z&=72. \end{aligned}$$

Whence, by subtracting the former of these from the latter, we have $5z=20$, or $z=4$.

$$\text{Consequently, } y=\frac{72+17z}{20}=\frac{72+68}{20}=\frac{140}{20}=7,$$

$$\text{and } x=\frac{22-4y+3z}{2}=\frac{22-28+12}{2}=\frac{6}{2}=3.$$

3. Given $x+y+z=53$, $x+2y+3z=105$, and $x+3y+4z=134$, to find the values of x, y, and z.

Ans. $x=24$, $y=6$, and $z=23$.

4. Given $x+\frac{1}{2}y+\frac{1}{3}z=32$, $\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15$, and $\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12$, to find the values of x, y, and z.

Ans. $x=12$, $y=20$, $z=30$.

5. Given $7x+5y+2z=79$, $8x+7y+9z=122$, and $x+4y+5z=55$, to find the values of x, y, and z.

Ans. $x=4$, $y=9$, $z=3$.

6. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find the values of x, y, and z.

Ans. $x=\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}c$,

$y=\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}c$,

$z=-\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c$

MISCELLANEOUS QUESTIONS,

PRODUCING SIMPLE EQUATIONS.

The usual method of resolving algebraic questions is first to denote the quantities that are to be found by x, y, or some of the other final letters of the alphabet; then, having properly examined the state of the question, perform with these letters, and the known quantities, by means of the common signs, the same operations and reasonings that it would be necessary to make if the quantities were known, and it was required to verify them, and the conclusion will give the result sought.

Or, it is generally best, when it can be done, to denote only one of the unknown quantities by x, y, or z; and then to determine the expression for the others from the nature of the question; after which the same method of reasoning may be followed as above. And, in certain cases, the substituting for the sums and differences of quantities, or other methods of proceeding, may be used; which practice and experience alone can suggest.

1. What number is that whose third part exceeds its fourth part by 16?

Let $x =$ the number required.

Then its $\frac{1}{3}$ part will be $\frac{1}{3}x$, and its $\frac{1}{4}$ part $\frac{1}{4}x$,

and therefore $\frac{1}{3}x-\frac{1}{4}x=16$, by the question;

that is, $x-\frac{3}{4}x=48$, or $4x-3x=192$.

Hence, $x=192$, the number required.

2. It is required to find two numbers, such, that their sum shall be 40, and their difference 16.

Let x denote the less of the two numbers required,
then will $x+16=$ the greater number,
and $x+x+16=40$, by the question.

That is, $2x=40-16$, or $x=\frac{24}{2}=12=$ less number,

and $x+16=12+16=28=$ the greater number required.

3. Divide 1000l. between A, B, and C, so that A shall have 72l. more than B, and C 100l. more than A.

Let $x=$ B's share of the given sum,
then will $x+72=$ A's share
and $x+172=$ C's share.

Hence their sum is $x+x+72+x+172=3x+244$;
$\therefore 3x+244=1000$, by the question
or $3x=1000-244=756$,

$$\therefore x=\frac{756}{3}=252l. = \text{B's share.}$$

Hence $x+72=324l.$ A's share,
$x+172=424l.$ C's share,
$252l.$ B's share.

Sum of all $=1000l.$ the proof.

4. It is required to divide 1000l. between two persons, so that their shares of it shall be in the proportion of 7 to 9.

Let $x=$ the first person's share,
then will $1000-x=$ second person's share.
$\therefore x : 1000-x :: 7 : 9$, by the question,
that is, $9x=(1000-x)\times 7=7000-7x$,

or $9x+7x=7000$, or $x=\frac{7000}{16}=437l.\ 10s.=$ 1st share,

and $1000-x=1000-437l.\ 10s.=562l.\ 10s.=$ 2d share.

5. The paving of a square court with stones, at 2s. a yard, will cost as much as the inclosing it with palisades at 5s. a yard; required the side of the square?

Let $x=$ length of the side of the square sought.
Then $4x=$ number of yards of inclosure,
and $x^2=$ number of yards of pavement.

Hence $4x\times5=20x=$ price of inclosing it,
and $x^2\times2=2x^2=$ the price of the paving.
Therefore $2x^2=20x$, by the question.
and, by division, $x=10$ yards, the length of the required side.

6. Out of a cask of wine, which had leaked away a third part, 21 gallons were afterwards drawn, and the cask being then gauged, appeared to be half full; how much did it hold?

Let $x=$ the number of gallons the cask is supposed to have held,

then the quantity leaked away $=\frac{1}{3}x$ gallons.

Whence the cask had lost, altogether, $21+\frac{1}{3}x$ gallons.

And therefore $21+\frac{1}{3}x=\frac{1}{2}x$, by the question.

That is, $63+x=\frac{3}{2}x$, or $126+2x=3x$,

consequently $3x-2x=126$, or $x=126$, the number of gallons required.

7. What fraction is that, to the numerator of which if 1 be added, its value will be $\frac{1}{3}$, but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction required be represented by $\frac{x}{y}$,

then $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$, by the question.

Hence $3x+3=y$, and $4x=y+1$, or $y=4x-1$,
therefore $3x+3=4x-1$, or $x=4$,
and $y=4x-1=15$.

Whence the fraction that was to be found is $\frac{4}{15}$.

8. A market woman bought a certain number of eggs at 2 a penny and as many others at 3 a penny, and

having sold them again, altogether, at the rate of 5 for $2d.$, found she had lost $4d.$; how many eggs had she?

Let $x =$ the number of eggs of each sort,

then will $\frac{1}{2}x =$ the price of the first sort,

and $\frac{1}{3}x =$ the price of the second sort.

Also, as $5 : 2x :: 2 : \frac{4x}{5}$ the price of both sorts, when mixed together, at the rate of 5 for $2d.$

Consequently $\frac{1}{2}x + \frac{1}{3}x - \frac{4x}{5} = 4$, by the question, that is, $15x + 10x - 24x = 120$, or $x = 120$, the number of eggs of each sort, as required.

9. If A can perform a piece of work in 10 days, and B in 13; in what time will they finish it, if they are both set about it together?

Let the time sought be denoted by x,

Then $\frac{1}{10} =$ the part done by A in one day,

$\frac{1}{13} =$ the part done by B in one day,

And $\frac{1}{x} =$ the part done by both in one day.

Consequently $\left(\frac{1}{10} + \frac{1}{13}\right) = \frac{1}{x}$,

$\therefore 13x + 10x = 130$, or $23x = 130$.

Whence $x = \frac{130}{23} = 5\frac{15}{23}$ days, the time required.

10. If one agent, A, alone can produce an effect e in the time a, and another agent, B, alone in the time b; in what time will both of them together produce the same effect?

Let the time sought be denoted by x,

Then $a : e :: x : \frac{ex}{a} =$ part of the effect produced by A,

and $b : e :: x : \frac{ex}{b} =$ part of the effect produced by B.

Hence $\frac{ex}{a} + \frac{ex}{b} = e$ (the whole effect) by the question.

Or $\frac{x}{a} + \frac{x}{b} = 1$ by dividing each side by e.

Therefore $x + \frac{ax}{b} = a$, or $bx + ax = ab$,

consequently $x = \frac{ab}{a+b} =$ the time required.

11. How much rye at 4*s.* 6*d.* a bushel must be mixed with 50 bushels of wheat at 6*s.* a bushel, so that the mixture may be worth 5*s.* a bushel?

Let $x =$ the number of bushels required,
then $9x$ is the price of the rye in sixpences,
and 600 the price of the wheat in ditto,
also $(50+x) \times 10$ the price of the mixture in ditto.
Whence $9x + 600 = 500 + 10x$, by the question,
or, by transposition, $10x - 9x = 600 - 500$.
Consequently $x = 100$, the number of bushels required.

12. A labourer engaged to serve for 40 days, on condition that for every day he worked he should receive 20*d.*, but for every day he was absent he should forfeit 8*d.*: now, at the end of the time, he had to receive 1*l.* 11*s.* 8*d.*; how many days did he work, and how many was he idle?

Let the number of days that he worked be denoted by x,
then will $40 - x$ be the number of days he was idle;
also $20x$ the sum earned, and $(40 - x) \times 8 = 320 - 8x$, the sum forfeited,

Whence $20x - (320 - 8x) = 380d.$ $(= 1l.\ 11s.\ 8d.)$, by the question,

that is, $20x - 320 + 8x = 380$,
or $28x = 380 + 320 = 700$.

Consequently $x = \frac{700}{28} = 25$, the number of days he worked, and $40 - x = 40 - 25 = 15$, the number of days he was idle.

QUESTIONS FOR PRACTICE.

1. It is required to divide a line of 15 inches in length, into two such parts, that one of them may be three-fourths of the other. *Ans.* $8\frac{4}{7}$ and $6\frac{3}{7}$.

2. My purse and money together are worth 20*s.*, and the money is worth 7 times as much as the purse; how much is there in it? *Ans.* 17*s.* 6*d.*

3. A shepherd, being asked how many sheep he had in his flock, said, If I had as many more, half as many more, and 7 sheep and a half, I should have just 500; how many had he? *Ans.* 197.

4. A post is one-fourth of its length in the mud, one-third in the water, and 10 feet above the water; what is its whole length? *Ans.* 24 feet.

5. After paying away a fourth of my money, and then a fifth of the remainder, I had 72 guineas left; what had I at first? *Ans.* 120 guineas.

6. It is required to divide 300*l.* between A, B, and C, so that A may have twice as much as B, and C as much as A and B together.

Ans. A 100*l.*, B 50*l.*, and C 150*l.*

7. A person, at the time he was married, was 3 times as old as his wife; but after they had lived together 15 years, he was only twice as old; what were their ages on their wedding day?

Ans. Bride's age 15, Bridegroom's 45.

8. It is required to find a number such, that if 5 be subtracted from it, two-thirds of the remainder shall be 40?

Ans. 65.

9. At a certain election, 1296 persons voted, and the successful candidate had a majority of 120; how many voted for each?

Ans. 708 for one, and 588 for the other.

10. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's 84, B's 42, and C's 14.

11. Two persons, A and B, lay out equal sums of money in trade; A gains 126*l.*, and B loses 87*l.*, and A's money is now double of B's; what did each lay out? *Ans.* 300*l.*

12. A person bought a chaise, horse, and harness, for 60*l.*; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each? *Ans.* 13*l.* 6*s.* 8*d.* for the horse, 6*l.* 13*s.* 4*d.* for the harness, and 40*l.* for the chaise.

13. A person was desirous of giving 3*d.* apiece to some beggars, but found he had not money enough in his pocket by 8*d.*, he therefore gave them each 2*d.*, and had then 3*d.* remaining; required the number of beggars? *Ans.* 11.

14. A servant agreed to live with his master for 8*l.* a year and a livery, but was turned away at the end of seven months, and received only 2*l.* 13*s.* 4*d.*, and his livery; what was its value? *Ans.* 4*l.* 16*s.*

15. A person left 560*l.* between his son and daughter, in such a manner, that for every half-crown the son should have, the daughter was to have a shilling; what were their respective shares? *Ans.* Son 400*l.*, Daughter 160*l.*

16. There is a certain number, consisting of two places of figures, which is equal to four times the sum of its digits; and if 18 be added to it, the digits will be inverted; what is the number? *Ans.* 24.

17. Two persons, A and B, have both the same income; A saves a fifth of his yearly, but B, by spending 50*l.* per annum more than A, at the end of four years finds himself 100*l.* in debt; what was their income? *Ans.* 125*l.*

18. When a company at a tavern came to pay their reckoning, they found, that if there had been three persons more, they would have had a shilling apiece less to pay, and if there had been two less, they would have had a shilling apiece more to pay; required the number of persons, and the quota of each?

Ans. 12 persons, quota of each 5*s.*

19. A person at a tavern borrowed as much money as he had about him, and out of the whole spent 1*s.*; he then went to a second tavern, where he also borrowed as much as he had now about him, and out of the whole spent 1*s.*; and going on, in this manner, to a third and fourth tavern, he found, after spending his shilling at the latter, that he had nothing left; how much money had he at first?

Ans. $11\frac{1}{4}d.$

20. It is required to divide the number 75 into two such parts, that three times the greater shall exceed seven times the less by 15. *Ans.* 54 and 21.

21. In a mixture of British spirits and water, one half of the whole plus 25 gallons was spirits, and a third part minus 5 gallons was water; how many gallons were there of each? *Ans.* 85 of spirits, and 35 of water.

22. A bill of 120*l.* was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; how many were there of each, reckoning the guinea at 21*s.* and the moidore at 27*s.*? *Ans.* 50.

23. Two travellers set out at the same time from London and York, whose distance from each other is 197 miles; one of them goes 14 miles a day, and the other 16; in what time will they meet? *Ans.* 6 days $13\frac{3}{5}$ hours.

24. There is a fish whose tail weighs 9*lb.*, his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish? *Ans.* 72*lb.*

25. It is required to divide the number 10 into three such parts, that, if the first be multiplied by 2, the second by 3, and the third by 4, the three products shall be all equal. *Ans.* $4\frac{8}{13}$, $3\frac{1}{13}$, and $2\frac{4}{13}$.

26. It is required to divide the number 36 into three such parts, that half the first, a third of the second, and a fourth of the third, shall be all equal to each other.

Ans. The parts are 8, 12, and 16.

27. A person has a saddle worth 50*l.*, which being put on the back of one of his two best horses, will make his value double that of the other, and if it be put on the back of the latter, it will make his value triple that of the former; what is the value of each horse?

Ans. One 30*l.*, and the other 40*l.*

28. A, in playing at billiards with B, won 5*s.* of him, and had then twice as much money as B had left; but B, in winning back his own money and 5*s.* more, had now three times as much as A had left; how much had each at first?

Ans. A 11*s.* and B 13*s.*

29. What two numbers are those whose difference, sum, and product, are to each other as the numbers 2, 3, and 5, respectively? *Ans.* 10 and 2.

30. A person in play lost a fourth of his money, and then won back 3*s*., after which he lost a third of what he now had, and then won back 2*s*.; lastly, he lost a seventh of what he then had, and after this found he had but 12*s*. remaining; what had he at first? *Ans*. 20*s*.

31. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but two of the greyhound's leaps are as much as three of the hare's; how many leaps must the greyhound take to catch the hare? *Ans*. 300.

32. It is required to divide the number 90 into four such parts, that if the first part be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient, shall be all equal.

Ans. The parts are 18, 22, 10, and 40.

33. A person, after spending 10*l*. more than a third of his yearly income, found he had 15*l*. more than half of it remaining; what was his income? *Ans*. 150*l*.

34. A man and his wife usually drank out a cask of beer in 12 days, but when the man was from home, it lasted the woman 30 days; how many days would the man alone be in drinking it? *Ans*. 20 days.

35. A general, ranging a division of his army in the form of a solid square, finds he has 34 men to spare, but increasing the side by one man, he wants 59 to fill up the square; of how many soldiers did the division consist?

Ans. 2150.

36. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days how many days will it take each person to perform the same work alone?

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

QUADRATIC EQUATIONS.

A QUADRATIC EQUATION, as before observed, is that in which the unknown quantity is of two dimensions, or which rises to the second power; and is either simple or compound.

A simple quadratic equation is that which contains only the square, or second power, of the unknown quantity; as

$$ax^2=b, \text{ or } x^2=\frac{b}{a}; \text{ where } x=\sqrt{\frac{b}{a}}.$$

A compound quadratic equation is that which contains both the first and second power of the unknown quantity; as

$$ax^2+bx=c, \text{ or } x^2+\frac{b}{a}x=\frac{c}{a}.$$

In which case, it is to be observed, that every equation of this kind, having any real positive root, will fall under one or other of the three following forms:

1. $x^2+ax=b$ . . . where $x=-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}$

2. $x^2-ax=b$ . . . where $x=+\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}$

3. $x^2-ax=-b$ . . . where $x=+\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}-b\right)}$.

Or, if the second and last terms be taken either positively or negatively, as they may happen to be, the general equation

$$ax^2\pm bx=\pm c, \text{ or } x^2\pm\frac{b}{a}x=\pm\frac{c}{a}$$

which comprehends all the three cases above mentioned, may be resolved by means of the following rule:

RULE.

Transpose all the terms that involve the unknown quantity to one side of the equation, and the known terms to the other; observing to arrange them so, that the term which contains the square of the unknown quantity may be positive, and stand first in the equation.

Then, if this square has any coefficient prefixed to it, let all the rest of the terms be divided by it, and the equation will be brought to one of the three forms above mentioned.

In which case the value of the unknown quantity x is always equal to half the coefficient or multiplier of x, in

the second term of the equation, taken with a contrary sign, together with $\pm$ the square root of the square of this number, and the known quantity that forms the absolute, or third term of the equation.*

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* This rule, which is more commodious in its practical application than that usually given, is founded upon the same principle; being derived from the well-known property, that in any quadratic equation

$$x^2 \pm ax = \pm b$$

if the square of half the coefficient a of the second term of the equation be added to each of its sides, so as to render it of the form

$$x^2 \pm ax + \tfrac{1}{4}a^2 = \tfrac{1}{4}a^2 \pm b,$$

that side which contains the unknown quantity will then be a complete square: and consequently, by extracting the root of each side, we shall have

$$x \pm \tfrac{1}{2}a = \pm\sqrt{(\tfrac{1}{4}a^2 \pm b)}, \text{ or } x = \mp\tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{4}a^2 \pm b)},$$

which is the same as the rule, taking a and b $+$ or $-$ as they may happen to be.

It may here, also, be observed, that the ambiguous sign $\pm$, which denotes both $+$ and $-$, is prefixed to the radical part of the value of x in every expression of this kind, because the square root of any positive quantity, as a^2, is either $+a$, or $-a$; for $(+a)\times(+a)$ or $(-a)\times(-a)$ are each $= +a^2$: but the square root of a negative quantity, as $-a^2$, is imaginary or unassignable, there being no quantity, either positive or negative, that, when multiplied by itself, will give a negative product.

To this we may also further add, that from the constant occurrence of the double sign before the radical part of the above expression, it necessarily follows, that every quadratic equation must have two roots; which are either both real, or both imaginary, according to the nature of the question.

Although we have said above that every quadratic equation has two roots, yet the young student must not on that account suppose that every question leading to a quadratic equation has two answers. For the question may be of such a nature as to render one of the two answers inadmissible, and which of them in such a case is to be rejected will always be pointed out by the conditions of the question itself. To explain this matter more fully, let us take an example.—Suppose that I sell goods for £56 and gain as much per cent. as the whole cost me, and that I required to know how much they stood me in, then the algebraical operation will be as follows:

Suppose b goods cost x pounds,
then the gain was $56-x$,
and by the question $100 : x :: x : 56-x$ . . . . . . (A)
$\therefore x^2 = 5600 - 100x$ or $x^2 + 100x = 5600$;
then by the rule $x = -50 \pm \sqrt{(50^2+5600)}$,
$\therefore x = -50 \pm \sqrt{(2500+5600)} = -50 \pm \sqrt{8100}$
$= -50 \pm 90$;
that is $x = 40$ or -140.

Now, here are two answers from which I know that b goods might

Note. All equations, which have the index of the unknown quantity, in one of their terms, just double that of the other, are resolved like quadratics, by first finding the value of the square root of the first term, according to the method used in the above rule, and then taking such a root, or power of the result, as is denoted by the reduced index of the unknown quantity.

Thus, if there be taken any general equation of this kind, as

$$x^{2m}+ax^m=b,$$

we shall have, by taking the square root of x^{2m}, and observing the latter part of the rule,

$$x^m=-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}, \text{ and } x=\left\{-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}\right\}^{\frac{1}{m}}.$$

And if the equation, which is to be resolved, be of the following form,

$$x^m-ax^{\frac{m}{2}}=b,$$

we shall necessarily have, according to the same principle,

$$x^{\frac{m}{2}}=\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}, \text{ and } x=\left\{\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}\right\}^{\frac{2}{m}}.$$

EXAMPLES.

1. Given $x^2+4x=140$, to find the value of x.

Here $x^2+4x=140$, by the question,

Whence $x=-2\pm\sqrt{(4+140)}$, by the rule,

Or, which is the same thing, $x=-2\pm\sqrt{144}$.

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have cost £40, or they might have cost £-140, but the latter result being negative is obviously inadmissible, therefore the true and only answer is £40.

We readily see the reason why we have obtained an answer more than can be admitted, for when the conditions of the question are transposed into symbols, they give us the proportion (A), and therefore we ought to get for x every possible value that will satisfy this proportion, whether the question itself admits every such value or not. If the question had been this, *viz.* To find a number such that it bears to £100 the same proportion that its deficiency from £56 bears to itself, we should have had the same algebraical condition (A), and both answers would have been admissible. It is the beauty of algebra, as this simple illustration may help to show, that it not only furnishes the proper solution to any given question, but gives, moreover, every solution that can possibly satisfy the same algebraical conditions.

Wherefore $x=-2+12=10$, or $-2-12=-14$, where one of the values of x is positive, and the other negative.

2. Given $x^2-12x+30=3$, to find the value of x.
Here $x^2-12x=3-30=-27$, by transposition
Whence $x=6\pm\sqrt{(36-27)}$, by the rule,
or, which is the same thing, $x=6\pm\sqrt{9}$
therefore $x=6+3=9$; or $=6-3=3$,
where it appears that x has two positive values.

3. Given $2x^2+8x-20=70$, to find the value of x.
Here $2x^2+8x=70+20=90$, by transposition,
and $x^2+4x=45$, by dividing by 2,
whence $x=-2\pm\sqrt{(4+45)}$, by the rule,
or, which is the same thing, $x=-2\pm\sqrt{49}$,
therefore $x=-2+7=5$, or $=-2-7=-9$,
where one of the values of x is positive and the other negative.

4. Given $3x^2-3x+6=5\frac{1}{3}$, to find the value of x.

Here $3x^2-3x=5\frac{1}{3}-6=-\frac{2}{3}$ by transposition,

and $x^2-x=-\frac{2}{9}$ by dividing by 3.

Whence $x=\frac{1}{2}\pm\sqrt{\left(\frac{1}{4}-\frac{2}{9}\right)}$ by the rule,

or, by subtracting $\frac{2}{9}$ from $\frac{1}{4}$, $x=\frac{1}{2}\pm\sqrt{\frac{1}{36}}$,

therefore $x=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$, or $=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$.

In which case x has two positive values.

5. Given $\frac{1}{2}x^2-\frac{1}{3}x+20\frac{1}{2}=42\frac{2}{3}$, to find the value of x.

Here $\frac{1}{2}x^2-\frac{1}{3}x=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$, by transposition,

and $x^2-\frac{2}{3}x=44\frac{1}{3}$, dividing by $\frac{1}{2}$ or multiplying by 2.

Whence we have $x=\frac{1}{3}\pm\sqrt{\left(\frac{1}{9}+44\frac{1}{3}\right)}$, by the rule.

Or, by adding $\frac{1}{9}$ and $44\frac{1}{3}$ together, $x=\frac{1}{3}\pm\sqrt{\frac{400}{9}}$

therefore $x=\frac{1}{3}+6\frac{2}{3}=7$, or $=\frac{1}{3}-6\frac{2}{3}=-6\frac{1}{3}$.

Where one value of x is positive, and the other negative.

6. Given $ax^2+bx=c$, to find the value of x.

Here $x^2+\frac{b}{a}x=\frac{c}{a}$ by dividing each side by a.

Whence, by the rule, $x=-\frac{b}{2a}\pm\sqrt{\left(\frac{b}{4a^2}+\frac{c}{a}\right)}$,

or, reducing the part within the radical

$$x=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{(b^2+4ac)}.$$

7. Given $ax^2-bx+c=d$, to find the value of x.

Here $ax^2-bx=d-c$, by transposition,

and $x^2-\frac{b}{a}x=\frac{d-c}{a}$, by dividing by a.

Whence $x=\frac{b}{2a}\pm\sqrt{\left(\frac{d-c}{a}+\frac{b^2}{4a^2}\right)}$ by the rule,

or, mult^s. $d-c$ and a by $4a$, $x=\frac{b}{2a}\pm\frac{1}{2a}\sqrt{\{4a(d-c)+b^2\}}$.

8. Given $x^4+ax^2=b$, to find the value of x.

Here, $x^2=-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}=-\frac{a}{2}\pm\frac{1}{2}\sqrt{(a^2+4b)}$, by the rule.

Whence $x=\pm\sqrt{\{-\frac{a}{2}\pm\frac{1}{2}\sqrt{(a^2+4b)}\}}$ by extracting the root on each side.

9. Given $\frac{1}{2}x^6-\frac{1}{4}x^3=-\frac{1}{32}$ to find the value of x.

By multiplying by 2, $x^6-\frac{1}{2}x^3=-\frac{1}{16}$,

whence $x^3=\frac{1}{4}\pm\sqrt{\left(\frac{1}{16}-\frac{1}{16}\right)}=\frac{1}{4}$ by the rule.

And consequently $x=\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\frac{1}{2}\sqrt[3]{2}$.

10. Given $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, to find the value of x.

By dividing by 2, $x^{\frac{2}{3}}+\frac{3}{2}x^{\frac{1}{3}}=1$,

whence $x^{\frac{1}{3}}=-\frac{3}{4}\pm\sqrt{\left(\frac{9}{16}+1\right)}=-\frac{3}{4}\pm\frac{5}{4}=\frac{1}{2}$ or -2.

Therefore $x=\left(\frac{1}{2}\right)^3=\frac{1}{8}$, or $(-2)^3=-8$.

11. Given $x^4-12x^3+44x^2-48x=9009$, to find the value of x.

This equation may be expressed as follows,

$$(x^2-6x)^2+8(x^2-6x)=9009,$$

whence, by the rule, $x^2-6x=-4\pm\sqrt{(16+9009)}$,

or $x^2-6x=91$ or -99.

The quadratic $x^2-6x=91$, gives

$$x=3\pm\sqrt{(9+91)}=3\pm10.$$

And the quadratic $x^2-6x=-99$, gives

$$x=3\pm\sqrt{(9-99)}=3\pm\sqrt{-90}.$$

Whence the four roots are 13, -7, $3+\sqrt{-90}$, and $3-\sqrt{-90}$.

EXAMPLES FOR PRACTICE.

1. Given $x^2-8x+10=19$, to find the values of x.
Ans. $x=9$ or -1.

2. Given $x^2-x-40=170$, to find the values of x.
Ans. $x=15$ or -14.

3. Given $3x^2+2x-9=76$, to find the values of x.
Ans. $x=5$ or $-\frac{17}{3}$.

4. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$, to find the values of x.

Ans. $x=1\frac{1}{2}$ or $-\frac{5}{6}$.

5. Given $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$, to find the values of x.

Ans. $x=49$ or $\frac{361}{9}$.

6. Given $x+\sqrt{(5x+10)}=8$, to find the values of x.

Ans. $x=18$ or 3.

7. Given $\sqrt{(10+x)}-\sqrt[4]{(10+x)}=2$, to find the values of x.

Ans. $x=6$ or -9.

8. Given $2x^4-x^2+96=99$, to determine the values of x.

Ans. $x=\frac{1}{2}\sqrt{6}$ or $\sqrt{-1}$.

9. Given $x^6+20x^3-10=59$, to find the values of x.

Ans. $x=\sqrt[3]{3}$ or $\sqrt[3]{-23}$.

10. Given $3x^{2n}-2x^n+3=11$, to find the values of x.

Ans. $x=\sqrt[n]{2}$ or $\sqrt[n]{-\frac{4}{3}}$.

11. Given $5\sqrt[4]{x}-3\sqrt{x}=1\frac{1}{3}$, to determine the values of x.

Ans. $3\frac{13}{81}$ or $\frac{1}{81}$.

12. Given $\frac{2}{3}x\sqrt{(3+2x^2)}=\frac{1}{2}+\frac{2}{3}x^2$ to determine the values of x.

Ans. $x=\frac{1}{2}\sqrt{(-3\pm3\sqrt{2})}$.

13. Given $x\sqrt{(\frac{6}{x}-x)}=\frac{1+x^2}{\sqrt{x}}$, to determine the values of x.

Ans. $x=(1\pm\frac{1}{2}\sqrt{2})^{\frac{1}{2}}$.

14. Given $\frac{1}{x}\sqrt{(1-x^3)}=x^2$, to determine the values of x.

Ans. $x=\left(-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}\right)^{\frac{1}{3}}$

15. Given $x\surd(\frac{a}{x} - 1) = \surd(x^2 - b^2)$, to determine the values of x. *Ans.* $x = \frac{1}{4}a \pm \frac{1}{4}\surd(a^2 + 8b^2)$.

16. Given $\sqrt{1 + x - x^2} - 2(1 + x - x^2) = \frac{1}{9}$, to determine the values of x.

Ans. $x = \frac{1}{2} \pm \frac{1}{6}\surd 41$, or $x = \frac{1}{2} \pm \frac{1}{3}\surd 11$.

17. Given $\surd\left(x - \frac{1}{x}\right) + \surd\left(1 - \frac{1}{x}\right) = x$, to determine the values of x. *Ans.* $x = \frac{1}{2} \pm \frac{1}{2}\surd 5$.

18. Given $x^{4n} - 2x^{3n} + x^n = 6$, to find the value of x.

Ans. $x = \sqrt[n]{\tfrac{1}{2} \pm \tfrac{1}{2}\surd 13}$, or $\sqrt[n]{\tfrac{1}{2} \pm \tfrac{1}{2}\surd -7}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS

The methods of expressing the conditions of questions of this kind, and the consequent reduction of them till they are brought to a quadratic equation, involving only one unknown quantity and its square, are the same as those already given for simple equations.

1. To find two numbers such that their difference shall be 8, and their product 240.

Let x equal the less number,
then will $x + 8 =$ the greater,
and $x(x + 8) = x^2 + 8x = 240$, by the question.
Whence $x = -4 + \surd(16 + 240) = -4 + \surd 256$, by the common rule, before given.
Therefore $x = 16 - 4 = 12$, the less number,
and $x + 8 = 12 + 8 = 20$, the greater.

2. It is required to divide the number 60 into two such parts, that their product shall be 864.

Let $x =$ the greater part,
then will $60 - x =$ the less,

And $x(60-x)=60x-x^2=864$, by the question,
or by changing the signs on both sides of the equation,
$$x^2-60x=-864.$$

Whence $x=30+\sqrt{(900-864)}=30+\sqrt{36}=36$, the greater part; and $60-x=60-36=24$, the less.

3. It is required to find two numbers such that their sum shall be $10(a)$, and the sum of their squares $58(b)$.

Let $x=$ the greater of the two numbers,
then will $a-x=$ the less,
and $x^2+(a-x)^2=2x^2-2ax+a^2=b$, by the question,
or $2x^2-2ax=b-a^2$, by transposition,
and $x^2-ax=\dfrac{b-a^2}{2}$, by division.

Whence $x=\dfrac{a}{2}+\sqrt{\left(\dfrac{a^2}{4}+\dfrac{b-a^2}{2}\right)}=\dfrac{a}{2}+\dfrac{1}{2}\sqrt{(2b-a^2)}$.

If 10 be put for a, and 58 for b, we shall have $x=$
$$\frac{10}{2}+\frac{1}{2}\sqrt{(116-100)}=7, \text{ the greater number,}$$
and $10-x=10-7=3$, the less.

4. Having sold a piece of cloth for $24l.$, I gained as much per cent. as it cost me; what was the price of the cloth?

Let $x=$ pounds the cloth cost,
then will $24-x=$ the whole gain.
But $100 : x :: x : 24-x$, by the question,
or $x^2=100(24-x)=2400-100x$,
that is, $x^2+100x=2400$,
whence $x=-50+\sqrt{(2500+2400)}=-50+70=20l.$,
the price of the cloth.

5. A person bought a number of sheep for $80l.$, and if he had bought 4 more for the same money, he would have paid $1l.$ less for each; how many did he buy?

Let x represent the number of sheep,
then will $\dfrac{80}{x}$ be the price of each,
and $\dfrac{80}{x+4}=$ price of each, if $x+4$ cost $80l.$

$$\text{But } \frac{80}{x}=\frac{80}{x+4}+1, \text{ by the question,}$$

$$\text{or } 80=\frac{80x}{x+4}+x, \text{ by multiplication,}$$

and $80x+320=80x+x^2+4x$, by the same,

or, by leaving out $80x$ on each side, $x^2+4x=320$.

Whence $x=-2+\sqrt{(4+320)}=-2+18=16$, the number of sheep.

6. It is required to find two numbers, such that the sum, product, and difference of their squares, shall be all equal to each other.

Let $x=$ the greater number, and $y=$ the less.

$$\text{Then } \left.\begin{cases} x+y=xy \\ x+y=x^2-y^2 \end{cases}\right\} \text{ by the question.}$$

$$\text{Hence } 1=\frac{x^2-y^2}{x+y}=x-y, \text{ or } x=y+1, \text{ by 2d equation,}$$

and $(y+1)+y=y(y+1)$ by 1st equation,

that is, $2y+1=y^2+y$; or $y^2-y=1$.

$$\text{Whence } y=\frac{1}{2}+\sqrt{\left(\frac{1}{4}+1\right)}=\frac{1}{2}+\frac{1}{2}\sqrt{5},$$

$$\text{and } x=y+1=\frac{3}{2}+\frac{1}{2}\sqrt{5}.$$

7. It is required to find four numbers in arithmetical progression, such that the product of the two extremes shall be 45, and the product of the means 77.

Let $x=$ less extreme, and $y=$ common difference, then x, $x+y$, $x+2y$, and $x+3y$, will be the four numbers.

$$\text{Hence } \left.\begin{cases} x(x+3y)=x^2+3xy=45 \\ (x+y)(x+2y)=x^2+3xy+2y^2=77 \end{cases}\right\} \text{ by the}$$

question,

and $2y^2=77-45=32$, by subtraction,

$$\text{or } y^2=\frac{32}{2}=16 \text{ by division, and } y=\sqrt{16}=4.$$

Therefore $x^2+3xy=x^2+12x=45$, by the 1st equation,

And consequently $x=-6+\sqrt{(36+45)}=-6+9=3$.

Whence the numbers are 3, 7, 11, and 15.

8. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

Let x, y, and z, be the three numbers,

then $xz=y^2$, by the nature of the progression.

And $\left.\begin{cases} x+y+z=14 \\ x^2+y^2+z^2=84 \end{cases}\right\}$ by the question,

hence $x+z=14-y$, by the second equation,

and $x^2+2xz+z^2=196-28y+y^2$, by squaring both sides,

or $x^2+z^2+2y^2=196-28y+y^2$, by putting $2y^2$ for its equal $2xz$,

and $\therefore x^2+y^2+z^2=196-28y$;

hence $196-28y=84$, by equality,

whence $y=\frac{196-84}{28}=4$, by transposition and division.

Again, by the 1st equation, $xz=y^2=16$, or $x=\frac{16}{z}$,

and $x+y+z=\frac{16}{z}+4+z=14$, by the 2d equation,

or $16+4z+z^2=14z$, or $z^2-10z=-16$.

Whence $x=5+\sqrt{(25-16)}=5+3=8$.

Therefore the three numbers are 8, 4, and 2.

9. It is required to find two numbers, such that their sum shall be 13 (a), and the sum of their fourth powers 4721 (b).

Let $x=$ the difference of the two numbers sought,

then will $\frac{1}{2}a+\frac{1}{2}x$, or $\frac{a+x}{2}$ the greater number,

and $\frac{1}{2}a-\frac{1}{2}x$, or $\frac{a-x}{2}=$ the less.

But $\frac{(a+x)^4}{16}+\frac{(a-x)^4}{16}=b$, by the question,

or $(a+x)^4+(a-x)^4=16b$, by multiplication,

or $2a^4+12a^2x^2+2x^4=16b$, by involution and addition,

and $x^4+6a^2x^2=8b-a^4$, by transposition and division.

Whence $x^2=-3a^2+\sqrt{(9a^4+8b-a^4)}=$
$-3a^2+\sqrt{(8a^4+b)}$, by the rule,

and $x=\sqrt{\{-3a^2+2\sqrt{2(a^4+b)}\}}$, by extracting the root, where, by substituting 13 for a, and 4721 for b, we shall have $x=3$.

$$\text{Therefore } \frac{13+x}{2}=\frac{16}{2}=8, \text{ the greater number,}$$

$$\text{and } \frac{13-x}{2}=\frac{10}{2}=5, \text{ the less number,}$$

the sum of which is 13, and $8^4+5^4=4721$.

QUESTIONS FOR PRACTICE.

1. It is required to divide the number 40 into two such parts, that the sum of their squares shall be 818.
Ans. 23 and 17.

2. To find a number such, that if you subtract it from 10, and then multiply the remainder by the number itself, the product shall be 21. *Ans.* 7 or 3.

3. It is required to divide the number 24 into two such parts, that their product shall be equal to 35 times their difference. *Ans.* 10 and 14.

4. It is required to divide the number 20 into two such parts, that twice the square of the greater part shall exceed three times the square of the less, by 96.
Ans. 12 and 8.

5. It is required to divide the number 60 into two such parts, that their product shall be to the sum of their squares in the ratio of 2 to 5. *Ans.* 20 and 40.

6. It is required to divide the number 146 into two such parts, that the difference of their square roots shall be 6. *Ans.* 25 and 121.

7. It is required to find two numbers, such that their sum shall be 23, and their product $116\frac{1}{4}$.
Ans. $7\frac{1}{2}$ and $15\frac{1}{2}$.

8. The sum of two numbers is $1\frac{1}{3}$, and the sum of their reciprocals $3\frac{1}{5}$; required the numbers. *Ans.* $\frac{1}{2}$ and $\frac{5}{6}$.

9. The difference of two numbers is 15, and half their product is equal to the cube of the less number; required the numbers. *Ans.* 3 and 18.

10. The difference of two numbers is 5, and the difference of their cubes 1685; required the numbers.

Ans. 8 and 13.

11. A person bought a quantity of cloth for 33*l.* 15*s.*, which he sold again at 2*l.* 8*s.* per piece, and gained by the bargain as much as one piece cost him; required the number of pieces. *Ans.* 15.

12. What two numbers are those, whose sum multiplied by the greater is equal to 77, and whose difference multiplied by the less is equal to 12? *Ans.* 4 and 7.

13. A grazier bought as many sheep as cost him 60*l.*, and after reserving 15 out of the number, sold the remainder for 54*l.*, and gained 2*s.* a head by them; how many sheep did he buy? *Ans.* 75.

14. It is required to find two numbers such, that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes. *Ans.* $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(\sqrt{5}+5)$.

15. The difference of two numbers is 8, and the difference of their fourth powers is 14560; required the numbers.

Ans. 3 and 11.

16. A company at a tavern had 8*l.* 15*s.* to pay for their reckoning; but, before the bill was settled, two of them left the house, in consequence of which, those who remained had 10*s.* apiece more to pay than before; how many were there in company? *Ans.* 7.

17. A person ordered 7*l.* 4*s.* to be distributed among some poor people; but, before the money was divided, there came in, unexpectedly, two claimants more, by which circumstance the former received a shilling apiece less than they would otherwise have done; what was the number of claimants at first? *Ans.* 16 persons.

18. It is required to find four numbers in geometrical progression such, that their sum shall be 15, and the sum of their squares 85. *Ans.* 1, 2, 4, and 8.

19. The sum of two numbers is 11, and the sum of their fifth powers is 17831; required the numbers.

Ans. 4 and 7.

20. It is required to find four numbers in arithmetical

progression such, that their common difference shall be 4, and their continued product 176985.

Ans. 15, 19, 23, and 27.

21. It is required to find two numbers such, that the square of the first plus their product shall be 140, and the square of the second minus their product 78.

Ans. 7 and 13.

22. Two detachments of foot being ordered to a station at the distance of 39 miles from their present quarters, began their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour faster than the other, arrived there an hour sooner; required their rates of marching.

Ans. $3\frac{1}{4}$ and 3 miles per hour.

OF CUBIC EQUATIONS.

A CUBIC equation is that in which the unknown quantity rises to three dimensions; and like quadratics, or those of the higher orders, is either simple or compound.

A simple cubic equation is of the form

$$ax^3=b, \text{ or } x^3=\frac{b}{a}; \text{ where } x=\sqrt[3]{\frac{b}{a}}.$$

A compound cubic equation is of the form

$$x^3+ax=b,\ x^3+ax^2=b, \text{ or } x^3+ax^2+bx=c,$$

in each of which the known quantities a, b, c, may be either $+$ or $-$.

Or, either of the two latter of these equations may be reduced to the same form as the first, by taking away its second term, which is done as follows:

RULE.

Take some new unknown quantity, and subjoin to it a third part of the coefficient of the second term of the equation with its sign changed; then if this sum, or difference, as it may happen to be, be substituted for the original unknown quantity and its powers in the proposed equation, there will arise an equation wanting its second term.

Note. The second term of any of the higher orders of equations may also be exterminated in a similar manner, by substituting for the unknown quantity the sum or difference, as above, of some other unknown quantity, and the 4th, 5th, &c., part of the coefficient of its second term, with its sign changed, according as the equation is of the 4th, 5th, &c., power.

EXAMPLES.

1. It is required to exterminate the second term of the equation $x^3+3ax^2=b$, or $x^3+3ax^2-b=0$.

$$\text{Here } x=z-\frac{3a}{3}=z-a,$$

$$\text{then}\begin{cases} x^3=z^3-3az^2+3a^2z-\ a^3 \\ 3ax^2=\quad +3az^2-6a^2z+3a^3 \\ -b= \qquad\qquad\qquad\quad -\ b \end{cases}$$

$$\text{Whence}\qquad z^3-3a^2z+2a^3-b=0,$$

$$\text{or } z^3-3a^2z=b-2a^3,$$

in which equation the second power (z^2) of the unknown quantity is wanting.

2. Let the equation $x^3-12x^2+3x=-16$ be transformed into another, that shall want the second term.

$$\text{Here } x=z+4,$$

$$\text{then}\begin{cases} (z+4)^3=z^3+12z^2+48z+\ 64 \\ -12(z+4)^2=\quad -12z^2-96z-192 \\ +\ 3(z+4)= \qquad\qquad +\ 3z+\ 12 \end{cases}$$

$$\text{Whence}\quad z^3-45z-116=-16$$

$$\text{or } z^3-45z=100,$$

which is an equation where z^2, or the second term, is wanting, as before.

3. Let the equation $x^3-6x^2=10$ be transformed into another, that shall want the second term.

Ans. $y^3-12y=26$.

4. Let $y^3-15y^2+81y=243$ be transformed into an equation that shall want the second term.

Ans. $x^3+6x=88$.

5. Let the equation $x^3+\frac{3}{4}x^2+\frac{7}{8}x-\frac{9}{16}=0$ be transformed into another, that shall want its second term.

Ans. $y^3+\frac{11}{16}y=\frac{3}{4}$

6. Let the equation $2x^3-3x^2+4x-5=0$ be transformed into another, that shall want its second term.

Ans. $z^3+\frac{5}{4}z=\frac{7}{4}$

OF THE SOLUTION OF CUBIC EQUATIONS.

RULE.

Take away the second term of the equation when necessary, as directed in the preceding rule. Then, if the numeral coefficients of the given equation, or of that arising from the reduction above mentioned, be substituted for a and b in either of the following formulæ, the result will give one of the roots required.

$$x^3+ax=b$$

$$x=\begin{cases}\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}+\sqrt[3]{\left\{\frac{b}{2}-\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}} \\ \text{or} \\ \sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}-\dfrac{\frac{1}{3}a}{\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}},\end{cases}$$

where it is to be observed, that when the coefficient a of the second term of the above equation is negative, $\frac{1}{3}a$, or its cube $\frac{a^3}{27}$, in the formula, will be negative; and if the absolute term b be negative, $\frac{b}{2}$, in the formula, will also be negative; but $\frac{b^2}{4}$ will be positive.*

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* This method of solving cubic equations is usually ascribed to CARDAN, a celebrated Italian analyst of the 16th century; but the

It may, likewise, be remarked, that when the equation is of the form $x^3 - ax = \pm b$

and $\frac{a^3}{27}$ is greater than $\frac{b^2}{4}$, or $4a^3$ greater than $27b^2$, the solution of it cannot be obtained by the above rule; as the question, in this instance, falls under what is usually called the *Irreducible Case* of cubic equations.*

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authors of it were SCIPIO FERREUS and NICHOLAS TARTALEA, who discovered it about the same time, independently of each other, as is proved by MONTUCLA, in his *Histoire des Mathématiques*, vol. i. p. 568, and more at large in HUTTON's *Mathematical Dictionary*, Art. Algebra.

The rule above given, which is similar to that of CARDAN, may be demonstrated as follows:—

Let the equation, whose root is required, be $x^3 + ax = b$.

And assume $y + z = x$, and $3yz = -a$.

Then, by substituting these values in the given equation, we shall have $y^3 + 3y^2z + 3yz^2 + z^3 + a \times (y+z) = y^3 + z^3 + 3yz \times (y+z) + a \times (y+z) = y^3 + z^3 - a \times (y+z) + a \times (y+z) = b$, or

$$y^3 + z^3 = b.$$

And if, from the square of this last equation, there be taken 4 times the cube of the equation $yz = -\frac{1}{3}a$, we shall have $y^6 - 2y^3z^3 + z^6 = b^2 + \frac{4}{27}a^3$, or

$$y^3 - z^3 = \sqrt{(b^2 + \tfrac{4}{27}a^3)}.$$

But the sum of this equation and $y^3 + z^3 = b$, is $2y^3 = b + \sqrt{(b^2 + \frac{4}{27}a^3)}$, and their difference is $2z^3 = b - \sqrt{(b^2 + \frac{4}{27}a^3)}$; whence $y = \sqrt[3]{\left\{\frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}\right\}}$, and $z = \sqrt[3]{\left\{\frac{1}{2}b - \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}\right\}}$.

From which it appears, that $y + z$, or its equal x, is =
$\sqrt[3]{\left\{\frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}\right\}} + \sqrt[3]{\left\{\frac{1}{2}b - \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}\right\}}$, which is the theorem above given.

Or, since z is $= -\frac{a}{3y}$, we shall have $y + z = y - \frac{a}{3y}$, or $x =$

$\sqrt[3]{\frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}} - \frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}}}$, being also the same as the rule.

* It may here be observed, as a remarkable circumstance in the history of this science, that the solution of the *Irreducible Case* above mentioned, has baffled the united efforts of the most celebrated mathematicians in Europe; although it is well known that all the three roots of the equation are, in this case, real; whereas, in those that are resolvable by the above formula, only one of the roots is real; so that, in fact, the rule is only applicable to such cubics as have two impossible roots. The solution of numerical equations of all orders has at length been effected by the recent discoveries of BUDAN, HORNER, and STURM, which PROFESSOR YOUNG, of Belfast, has admirably combined, both in their theoretical and practical bearings, in his recent work on *Algebraical Equations*, published by John Souter, 73, St. Paul's Church-yard.

EXAMPLES.

1. Given $2x^3-12x^2+36x=44$, to find the value of x.

Here $x^3-6x^2+18x=22$, by dividing by 2.

And, in order to exterminate the second term,

$$\text{put } x=z+\frac{6}{3}=z+2,$$

$$\text{then } \left|\begin{array}{rl} (z+2)^3= & z^3+6z^2+12z+\ 8 \\ -6(z+2)^2= & \quad -6z^2-24z-24 \\ 18(z+2)= & \qquad\qquad 18z+36 \end{array}\right| =22.$$

Whence $z^3+6z+20=22$, or $z^3+6z=2$.

And consequently, by substituting 6 for a, and 2 for b, in the first formula, we shall have

$$x=\sqrt[3]{\left\{\frac{2}{2}+\sqrt{\left(\frac{4}{4}+\frac{216}{27}\right)}\right\}}+\sqrt[3]{\left\{\frac{2}{2}-\sqrt{\left(\frac{4}{4}+\frac{216}{27}\right)}\right\}}=$$

$$\sqrt[3]{\{1+\sqrt{(1+8)}\}}+\sqrt[3]{\{1-\sqrt{(1+8)}\}}=\sqrt[3]{(1+\sqrt{9})}+\sqrt[3]{(1-\sqrt{9})}=\sqrt[3]{(1+3)}+\sqrt[3]{(1-3)}=\sqrt[3]{4}-\sqrt[3]{2}.$$

Therefore $x=z+2=\sqrt[3]{4}-\sqrt[3]{2}+2=2+1\cdot587401-1\cdot259921=2\cdot32748$, the answer.

2. Given $x^3-6x=12$, to find the value of x.

Here a being equal to -6, and b equal to 12, we shall have, by the 2d formula,

$$x=\sqrt[3]{\{6+\sqrt{(36-8)}\}}-\frac{-2}{\sqrt[3]{\{6+\sqrt{(36-8)}\}}}=$$

$$\sqrt[3]{(6+\sqrt{28})}+\frac{2}{\sqrt[3]{(6+\sqrt{28})}}=\sqrt[3]{(6+5\cdot2915)}+$$

$$\frac{2}{\sqrt[3]{(6+5\cdot2915)}}=\sqrt[3]{(11\cdot2915)}+\frac{2}{\sqrt[3]{(11\cdot2915)}}=$$

$$2\cdot2435+\frac{2}{2\cdot2435}=2\cdot2435+\cdot8915=3\cdot135.$$

Therefore $x=3\cdot135$, the answer.

3. Given $x^3-2x=-4$, to find the value of x.

Here a being $=-2$, and $b=-4$, we shall have, by the formula,

$$x=\sqrt[3]{\left\{-2+\sqrt{\left(4-\frac{8}{27}\right)}\right\}}+\sqrt[3]{\left\{-2-\sqrt{\left(4-\frac{8}{27}\right)}\right\}},\ \text{or}$$

by reduction, $x = \sqrt[3]{(-2 + \frac{10}{9}\sqrt{3})} - \sqrt[3]{(2 + \frac{10}{9}\sqrt{3})} =$

$\sqrt[3]{(-2+1\cdot9245)} - \sqrt[3]{(2+1\cdot9245)} = \sqrt[3]{(-\cdot0755)} -$
$\sqrt[3]{(3\cdot9245)} = -\cdot4226 - 1\cdot5773 = -1\cdot9999$, or -2.

Therefore $x = -2$, the answer.*

Note. When one of the roots of a cubic equation has been found, either by trial or in any other way, the other two roots may be determined as follows:

Let the known root be denoted by r, and put all the terms of the equation, when brought to the left-hand side, $=0$; then if the equation so formed be divided by $x \mp r$, according as r is positive or negative, there will arise a quadratic equation, the roots of which will be the other two roots of the given cubic equation.

Thus, supposing $x^3 - 15x = 4$, we can readily find, by a few trials, that $x = 4$;

$$
\begin{array}{l}
x-4)\,x^3-15x-4\,(x^2+4x+1 \\
\quad\;\; \underline{x^3-4x^2} \\
\qquad\quad 4x^2-15x \\
\qquad\quad \underline{4x^2-16x} \\
\qquad\qquad\quad x-4 \\
\qquad\qquad\quad \underline{x-4} \\
\qquad\qquad\qquad *
\end{array}
$$

Whence, according to the note above given,

$$x^2 + 4x + 1 = 0, \text{ or } x^2 + 4x = -1;$$

the two roots of which quadratic are $-2+\sqrt{3}$ and $-2-\sqrt{3}$; and consequently

$$4, -2+\sqrt{3}, \text{ and } -2-\sqrt{3},$$

are the three roots of the proposed equation.

EXAMPLES FOR PRACTICE.

1. Given $x^3 + 3x^2 - 6x = 8$, to find the roots of the equation, or the values of x. *Ans.* $x = -1$, $+2$, or -4.

---

* When the root of the given equation is a whole number, this method determines it only by an approximation of 9s. in the decimal part, which sufficiently indicates the entire integer; but in most instances of this kind, its value may be more readily found by a few trials from the equation itself.

2. Given $x^3+x^2=500$, to find the root of the equation, or the value of x. *Ans.* $x=7\cdot6172$.

3. Given $x^3+12x=20$, to find the root of the equation, or the value of x. *Ans.* $x=1\cdot42535$.

4. Given $x^3-6x=6$, to find the root of the equation, or the value of x. *Ans.* $x=\sqrt[3]{2}+\sqrt[3]{4}$.

5. Given $x^3+9x=6$, to find the root of the equation, or the value of x. *Ans.* $x=\sqrt[3]{9}-\sqrt[3]{3}$.

6. Given $x^3-22x=24$, to find the root of the equation, or the value of x.

$$Ans.\ x=\sqrt[3]{\left(12+\frac{1}{3}\sqrt{-6760}\right)}+\sqrt[3]{\left(12-\frac{1}{3}\sqrt{-6760}\right)}.$$

7. Given $x^3-17x^2+54x=350$, to find the root of the equation, or the value of x. *Ans.* $x=14\cdot95429$.

OF BIQUADRATIC EQUATIONS.

A *biquadratic equation*, as before observed, is one that rises to the fourth power, or which is of the general form

$$x^4+ax^3+bx^2+cx+d=0.$$

Or, when its second term is taken away, of the form

$$x^4+bx^2+cx+d=0,$$

to which it can always be reduced; and, in that case, its solution may be obtained by the following rule:—

Find the value of z in the cubic equation

$$z^3-\left(\frac{1}{12}b^2+d\right)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd,$$

and let the root thus determined be denoted by r.

Then find the two values of x, in each of the following quadratic equations.

$$x^2+\sqrt{\{2(r-\frac{1}{3}b)\}}x=-(r+\frac{1}{6}b)+\sqrt{\{(r+\frac{1}{6}b)^2-d\}}$$

$$x^2-\sqrt{\{2(r-\frac{1}{3}b)\}}x=-(r+\frac{1}{6}b)-\sqrt{\{(r+\frac{1}{6}b)^2-d\}}$$

and they will be the four roots of the biquadratic equations required.*

---

* The method of solving biquadratic equations was first discovered by Louis Ferrars, a disciple of the celebrated Cardan, before men-

Or the four roots of the given equation, in this last case, will be as follows:

$$x=-\tfrac{1}{2}\sqrt{\{2(r-\frac{1}{3}b)\}}+\sqrt{\{-\frac{r}{2}+\frac{b}{3}+\sqrt{[(r+\frac{1}{6}b)^2-d]}\}}$$

$$x=-\tfrac{1}{2}\sqrt{\{2(r-\frac{1}{3}b)\}}-\sqrt{\{-\frac{r}{2}-\frac{b}{3}+\sqrt{[(r+\frac{1}{6}b)^2-d]}\}}$$

$$x=+\tfrac{1}{2}\sqrt{\{2(r-\frac{1}{3}b)\}}+\sqrt{\{-\frac{r}{2}-\frac{b}{3}-\sqrt{[(r+\frac{1}{6}b)^2-d]}\}}$$

$$x=+\tfrac{1}{2}\sqrt{\{2(r-\frac{1}{3}b)\}}-\sqrt{\{-\frac{r}{2}-\frac{b}{3}-\sqrt{[(r+\frac{1}{6}b)^2-d]}\}}$$

---

tioned; but the above rule is derived from that given by DESCARTES in his *Geometry*, published in 1637, the truth of which may be shown as follows:

Let the equation, which is to be resolved, be

$$x^4+ax^2+bx+c=0,$$

and conceive it to be produced by the multiplication of the two quadratics

$$x^2+px+q=0, \text{ and } x^2+rx+s=0.$$

Then, since these equations, as well as the given one, are each $=0$, there will arise, by taking their product,

$$x^4+(p+r)x^3+(s+q+pr)x^2+(ps+qr)x+qs=x^4+ax^2+bx+c.$$

And, consequently, by equating the homologous terms of this last equation, we shall have the four following equations,

$$p+r=0;\ s+q+pr=a;\ ps+qr=b;\ qs=c.$$

$$\text{or } r=-p,\ s+q=a+p^2,\ s-q=\frac{b}{p},\ qs=c.$$

Whence, subtracting the square of the third of these from that of the second, and then changing the sides of the equations, we shall have

$$a^2+2ap^2+p^4-\frac{b^2}{p^2}=4qs, \text{ or } 4c;\ \text{ or } p^6+2ap^4+(a^2-4c)p^2=b^2.$$

Where, putting $p^2=z$, the value of z, and consequently of p, may be found by the rule before given for cubic equations.

Hence, also, since $s+q=a+p^2$, and $s-q=\frac{b}{p}$, there will arise, by addition and subtraction,

$$s=\frac{1}{2}a+\frac{1}{2}p^2+\frac{b}{2p};\ q=\frac{1}{2}a+\frac{1}{2}p^2-\frac{b}{2p};$$

where p being known, the values of s and q are likewise known.

And, therefore, by extracting the roots of the two assumed quadratics $x^2+px+q=0$, and $x^2+rx+s=0$, or its equal $x^2-px+s=0$, we shall have

$$x=-\tfrac{1}{2}p\pm\sqrt{(\tfrac{1}{4}p^2-q)};\ x=\tfrac{1}{2}p\pm\sqrt{(\tfrac{1}{4}p^2-s)};$$

which expressions, when taken in $+$ and $-$, give the four roots of the proposed biquadratic, as was required.

It may be observed, that when p, in the cubic equation $p^6+2ap^4+(a^2-4c)p^2=b^2$, is rational, the question may be solved by quadratics.

EXAMPLES.

1. Given $x^4+12x-17=0$, to find the four roots of the equation.

Here $a=0$, $b=0$, $c=12$, and $d=-17$.

Whence, by substituting these numbers in the cubic equation

$$z^3-\left(\frac{1}{12}b^2+d\right)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd,$$

we shall have $z^3+17z=18$.

Where it is evident, by inspection, that $z=1$.

And if this number be substituted for r, 0 for b, and -17 for d in the two quadratic equations in the above rule, their solution will give

$$x=-\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}+\sqrt{18})}=-\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}+3\sqrt{2})}$$
$$x=-\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}+\sqrt{18})}=-\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}+3\sqrt{2})}$$
$$x=+\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}-\sqrt{18})}=+\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}-3\sqrt{2})}$$
$$x=+\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}-\sqrt{18})}=+\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}-3\sqrt{2})}$$

Which are the four roots of the proposed equation ; the two first being real, and the two last imaginary.

2. Given $x^4-55x^2-30x+504=0$, to find the four roots, or values of x.

Ans. 3, 7, -4, and -6.

3. Given $x^4+2x^3-7x^2-8x=-12$, to find the four roots, or values of x.

Ans. 1, 2, -3, and -2.

4. Given $x^4-8x^3+14x^2+4x=8$, to find the four roots, or values of x.

$$\textit{Ans.}\begin{cases}3+\sqrt{5},\ 3-\sqrt{5}\\ 1+\sqrt{3},\ 1-\sqrt{3}.\end{cases}$$

5. Given $x^4-17x^2-20x-6=0$, to find the four roots, or values of x.

$$Ans. \begin{cases} 2+\sqrt{7}, & 2-\sqrt{7} \\ -2+\sqrt{2}, & -2-\sqrt{2}. \end{cases}$$

6. Given $x^4-27x^3+162x^2+356x-1200=0$, to find the four roots of the equation.

$$Ans. \begin{cases} 2\cdot0561, & -3\cdot0000 \\ 13\cdot1531, & 14\cdot7909. \end{cases}$$

7. Given $x^4-12x^2+12x-3=0$, to find the four roots of the equation.

$$Ans. \begin{cases} 0\cdot606018, & -3\cdot907377 \\ 2\cdot858084, & 0\cdot443278. \end{cases}$$

OF THE

RESOLUTION OF EQUATIONS

BY APPROXIMATION.

Find, by trial, a number nearly equal to the root sought, which call r; and let z be made to denote the difference between this assumed root and the true root x.

Then, instead of x in the given equation, substitute its equal $r\pm z$, and there will arise a new equation involving only z and known quantities.

Reject all the terms of this equation in which z is of two or more dimensions; and the approximate value of z may then be determined by means of a simple equation.

And if the value, thus found, be added to, or subtracted from that of r, according as r was assumed too little or too great, it will give a near value of the root required.

But as this approximation will seldom be sufficiently exact, the operation must be repeated, by substituting the number thus found, for r in the abridged equation exhibiting the value of z; when a second correction of z will be obtained, which being added to, or subtracted from r, will give a nearer value of the root than the former.

And by again substituting this last number for r, in the above-mentioned equation, and repeating the same process as often as may be thought necessary, a value of x may be found to any degree of accuracy required.

Note. The decimal part of the root, as found both by this and the next rule, will, in general, about double itself at each operation; and therefore it would be useless, as well as troublesome, to use a much greater number of figures than these in the several substitutions for the values of r.*

EXAMPLES.

1. Given $x^3+x^2+x=90$, to find the value of x by approximation.

Here the root, as found by a few trials, is nearly equal to 4.

Let therefore $4=r$, and $r+z=x$.

$$\text{Then } \left|\begin{array}{l} x^3=r^3+3r^2z+3rz^2+z^3 \\ x^2=r^2+2rz+z^2 \\ x=r+z \end{array}\right| =90.$$

* It may here be observed, that if any of the roots of an equation be whole numbers, they may be determined by substituting 1, 2, 3, 4, &c., successively, both in *plus* and in *minus*, for the unknown quantity, till a result is obtained equal to that in the question; when those that are found to succeed will be the roots required.

Or, since the last term of any equation is always equal to the continued product of all its roots, the number of these trials may be generally diminished, by finding all the divisors of that term, and then substituting them both in *plus* and in *minus*, as before, for the unknown quantity; when those that give the proper result will be the rational roots sought; but if none of them are found to succeed, it may be concluded that the equation cannot be resolved by this method; the roots, in that case, being either irrational or imaginary.

And by rejecting the terms z^3, $3rz^2$, and z^2, as small in comparison with z, we shall have

$$r^3+r^2+r+3r^2z+2rz+z=90\,;$$

whence $z=\dfrac{90-r^3-r^2-r}{3r^2+2r+1}=\dfrac{90-64-16-4}{48+8+1}=\dfrac{6}{57}=\cdot 10$

And consequently $x=4\cdot 1$, *nearly*.

Again, if 4·1 be substituted in the place of r, in the last equation, we shall have

$$z=\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-68\cdot 921-16\cdot 81-4\cdot 1}{50\cdot 43+8\cdot 2+1}=\cdot 00283.$$

And consequently $x=4\cdot 1+\cdot 00283=41\cdot 0283$, for a *second approximation*.

And, if the first four figures, 4·102, of this number be again substituted for r, in the same equation, a still nearer value of the root will be obtained; and so on, as far as may be thought necessary.

2. Given $x^2+20x=100$, to find the value of x by approximation. *Ans.* $x=4\cdot 1421356$.

3. Given $x^3+9x^2+4x=80$, to find the value of x by approximation. *Ans.* $x=2\cdot 4721359$.

4. Given $x^4-38x^3+210x^2+538x+289=0$, to find the value of x by approximation. *Ans.* $x=30\cdot 53565375$.

5. Given $x^5+6x^4-10x^3-112x^2-207x+110=0$, to find the value of x by approximation.

Ans. 4·46410161.

The roots of equations, of all orders, can also be determined, to any degree of exactness, by means of the following easy rule of Double Position; which, though it has not been generally employed for this purpose, will be found, in some respects, superior to the former, as it can be applied, at once, to any unreduced equation, consisting of surds, or compound quantities, as readily as if it had been brought to its usual form.*

* Another method of approximating to the roots of equations will be found in the *Addenda*.

RULE.

Find, by trial, two numbers nearly equal to the root sought, and substitute them in the given equation instead of the unknown quantity, noting the results that are obtained from each.

Then, as the difference of these results is to the difference of the two assumed numbers, so is the difference between the true result, given by the question, and either of the former, to the correction of the number belonging to the result used; which correction being added to that number when it is too little, or subtracted from it when it is too great, will give the root required, *nearly*.

And if the number thus determined, and the nearest of the two former, or any other that appears to be more accurate, be now taken as the assumed roots, and the operation be repeated as before, a new value of the unknown quantity will be obtained still more correct than the first; and so on, proceeding in this manner, as far as may be judged necessary.*

---

* The above rule of Double Position, which is much more simple and commodious than the one commonly employed for this purpose, is the same as that which was first given at p. 311 of the octavo edition of my *Arithmetic*, published in 1810.

To this we may further add, that when one of the roots of an equation has been found, either by this method or the former, the other roots may be determined as follows:

Bring all the terms to the left-hand side of the equation, and divide the whole expression, so formed, by the difference between the unknown quantity (x) and the root first found; and the resulting equation will then be depressed a degree lower than the given one.

Find a root of this new equation, by approximation, as in the first instance, and the number so obtained will be a second root of the original equation.

Then, by means of this root, and the unknown quantity, depress this second equation a degree lower, and thence find a third root: and so on, till the equation is reduced to a quadratic; when the two roots of this, together with the former, will be the roots of the equation required.

Thus, in the equation $x^3-15x^2+63x=50$, the first root is found, by approximation, to be $1 \cdot 02804$. Hence

$$x-1 \cdot 02804)x^3-15x^2+63x-50(x^2-13 \cdot 97196x+48 \cdot 63627=0.$$

And

EXAMPLES.

1. Given $x^3+x^2+x=100$, to find an approximate value of x.

Here it is soon found, by a few trials, that the value of x lies between 4 and 5.

Hence, by taking these as the two assumed numbers, the operation will stand as follows:

First Sup.		*Second Sup.*
4	x	5
16	x^2	25
64	x^3	125
84	Results	155

Therefore

155	5	100	
84	4	84	
71 :	1 ::	16 :	·225.

And consequently $x=4+·225=4·225$, *nearly*.

Again, if 4·2 and 4·3 be now taken as the two assumed numbers, the operation will stand thus:

First Sup		*Second Sup.*
4·2	x	4·3
17·64	x^2	18·49
74·088	x^3	79·507
95·928	Results	102·297

Therefore

102·297	4·3	102·297	
95·928	4·2	100	
6·369 :	·1 ::	2·297 :	·036.

And consequently $x=4·3-·036=4·264$, *nearly*.

And the two roots of the quadratic equation, $x^2-13·97196x=-48·63627$, found in the usual way, are 6·57653 and 7·39543.

So that the three roots of the given cubic equation, $x^3-15x^2+63x=50$, are 1·02804, 6·57653, and 7·39543; their sum being $=15$, the coefficient of the second term of the equation, as it ought to be when they are right.

Again, let 4·264 and 4·265 be the two asumed numbers; then

First Sup.		*Second Sup.*
4·264	. . x . .	4·265
18·181696	. . x^2 . .	18·190225
77·526752	. . x^3 . .	77·581310
99·972448	Results	100·036535

Therefore

100·036535	4·265	100
99·972448	4·264	99·972448
·064087 :	·001 ::	·027552 : ·0004299.

And consequently

$x = 4{\cdot}264 + {\cdot}0004299 = 4{\cdot}2644299$, *very nearly.*

2. Given $(\frac{1}{5}x^2 - 15)^2 + x\sqrt{x} = 90$, to find an approximate value of x.

Here, by a few trials, it will be soon found that the value of x lies between 10 and 11; which let, therefore, be the two assumed numbers, agreeably to the directions given in the rule.

Then

	First Sup.		*Second Sup.*
	25	. . $(\frac{1}{5}x^2-15)^2$ . .	84·64
	31·623	. . $x\sqrt{x}$ . .	36·483
	56·623	Results	121·123
Hence	121·123	. . 11 . .	121·123
	56·623	. . 10 . .	90
	64·5 :	1 ::	31·123 : ·482.

And consequently $x = 11 - {\cdot}482 = 10{\cdot}518$.

Again, let 10·5 and 10·6 be the two assumed numbers,

Then

First Sup.		*Second Sup.*
49·7025 ..	$(\frac{1}{5}x^2-15)^2$ ..	55·830784
34·0239 ..	$x\sqrt{x}$ ..	34·511099
83·7264 ..	Results ..	90·341883

Hence

90·341883	..	10·6	..	90·341883	
83·7264	..	10·5	..	90·	
6·615483	:	·1	::	·341883 :	·0051679.

And consequently

$x = 10{\cdot}6 - {\cdot}0051679 = 10{\cdot}5948321$, *very nearly.*

EXAMPLES FOR PRACTICE.

1. Given $x^3+10x^2+5x=2600$, to find a near approximate value of x. *Ans.* $x=11{\cdot}00675$.

2. Given $2x^4-16x^3+40x^2-30x+1=0$, to find a near value of x. *Ans.* $x=1{\cdot}284724$.

3. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x. *Ans.* $8{\cdot}414455$.

4. Given $\sqrt[3]{(7x^3+4x^2)}+\sqrt{(20x^2-10x)}=28$, to find the value of x. *Ans.* $4{\cdot}510661$.

5. Given $\sqrt{\{144x^2-(x^2+20)^2\}}+\sqrt{\{196x^2-(x^2+24)^2\}}=114$, to find the value of x. *Ans.* $7{\cdot}123883$.

OF EXPONENTIAL EQUATIONS.

An exponential quantity is that which is to be raised to some unknown power, or which has a variable quantity for its index, as

$$a^x,\ a^{\frac{1}{x}},\ x^x, \text{ or } x^{\frac{1}{x}}, \text{ \&c.}$$

And an exponential equation is that which is formed between any expression of this kind and some other quantity, whose value is known; as

$$a^x=b,\ x^x=a, \text{ \&c.}$$

where it is to be observed, that the first of these equations, when converted into logarithms, is the same as

$$x \text{ log. } a = \text{log. } b, \text{ or } x = \frac{\text{log. } b}{\text{log. } a},$$

And the second $x^x=a$, is the same as

$$x \text{ log. } x = \text{log. } a.$$

In the latter of which cases a near approximate value of the unknown quantity may be determined, as follows:

RULE.

Find, by trial, two numbers as near as can conveniently be done to the number sought, and substitute them in the given equation,

$$x \text{ log. } x = \text{log. } a,$$

instead of the unknown quantity, noting the results obtained from each, as in the rule of Double Position, before laid down.

Then, by means of a certain number of successive operations, performed in the same manner as is there described, the value of x may be found to any degree of accuracy required*.

EXAMPLES.

1. Given $x^x = 100$, to find an approximate value of x.

Here, by the above formula, we have

$$x \text{ log. } x = \text{log. } 100 = 2.$$

And since x is readily found, by a few trials, to be nearly in the middle between 3 and 4, but rather nearer the latter than the former, let 3·5 and 3·6 be taken for the two assumed numbers.

Then log. 3·5 = ·5440680; which, being multiplied by 3·5, gives 1·904238 = first result.

And log. 3·6 = ·5563025; which, being multiplied by 3·6, gives 2·002689 for the second result.

Whence

$$\begin{array}{lclcl} 2\cdot002689 & . . & 3\cdot6 & . . & 2\cdot002689 \\ 1\cdot904238 & . . & 3\cdot5 & . . & 2\cdot \\ \hline \cdot098451 & : & \cdot1 & :: & \cdot002689 : \cdot00273 \end{array}$$

for the first correction; which, taken from 3·6, leaves $x = 3\cdot59727$, *nearly*.

---

* Many attempts have been made to determine the value of the unknown quantity, in the exponential equation $x^x = a$, above given, by converting it into a series, the terms of which shall consist only of a and its powers; but no expression of this kind has hitherto been discovered, which is sufficiently convergent to answer any practical purpose. See Vol. II. of my *Treatise on Algebra*, before referred to.

And as this value is found, by trial, to be rather too small, let 3·59727 and 3·59728 be taken as the two assumed numbers.

Then log. 3·59727=·5559731; which, being multiplied by 3·59727, gives 1·9999854 = first result.

And log. 3·59728=·5559743; which, being multiplied by 3·51728, gives 1·9999952 = second result.

Whence

1·9999952..3·59728..2·
1·9999854..3·59727..1·9999952

·0000098 : ·00001 :: ·0000048 : ·00000485

for the second correction; which, added to 3·59728, gives x=3·59728485, the answer required; being a value of x extremely near the truth.

2. Given $x^x=2000$, to find an approximate value of x.
Ans. $x=4{\cdot}82782263$.

3. Given $(6x)^x=96$, to find an approximate value of x.
Ans. $x=1{\cdot}8826432$.

4. Given $x^x=123456789$, to find an approximate value of x.
Ans. 8·6400268.

5. Given $x^x-x=(2x-x^x)^{\frac{1}{x}}$, to find an approximate value of x.
Ans. $x=1{\cdot}747933$.

OF THE

BINOMIAL THEOREM.

The binomial theorem is a general algebraical expression or formula, by which any power, or root of a given quantity, consisting of two terms, is expanded into a series, the form of which, as it was first proposed by Sir I. Newton, being as follows:

$$(\mathrm{P}+\mathrm{PQ})^{\frac{m}{n}}=\mathrm{P}^{\frac{m}{n}}\Big[1+\frac{m}{n}\mathrm{Q}+\frac{m}{n}\Big(\frac{m-n}{2n}\Big)\mathrm{Q}^2+\frac{m}{n}\Big(\frac{m-n}{2n}\Big)\Big(\frac{m-2n}{3n}\Big)\mathrm{Q}^3+\frac{m}{n}\Big(\frac{m-n}{2n}\Big)\Big(\frac{m-2n}{3n}\Big)\Big(\frac{m-3n}{4n}\Big)\mathrm{Q}^4\ \&\text{c.}\Big]$$

$$(\mathrm{P}+\mathrm{PQ})^{\frac{m}{n}}=\mathrm{P}^{\frac{m}{n}}\Big[+\frac{m}{n}\mathrm{AQ}+\frac{m-n}{2n}\mathrm{BQ}+\frac{m-2n}{3n}\mathrm{CQ}+\frac{m-3n}{4n}\mathrm{DQ}+\frac{m-4n}{5n}\mathrm{EQ},\ \&\text{c.}\Big]$$

Where P is the first term of the binomial, Q the second term divided by the first, $\frac{m}{n}$ the index of the power, or root; and A, B, C, &c., the terms immediately preceding those in which they are first found, including their signs + or −.

This theorem may be readily applied to any particular case, by substituting the numbers, or letters, in the given example, for P, Q, m, and n, in either of the above formulæ, and then finding the result according to the rule.*

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* This celebrated theorem, which is of the most extensive use in algebra, and various other branches of analysis, may be otherwise expressed as follows:

$$(a+x)^{\frac{m}{n}} = a^{\frac{m}{n}}\left[1+\frac{m}{n}\left(\frac{x}{a}\right)+\frac{m}{n}.\frac{m-n}{2n}\left(\frac{x}{a}\right)^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}\right.$$
$$\left.\times\left(\frac{x}{a}\right)^3+\text{\&c.}\right]$$

$$\text{Or } (a+x)^{\frac{m}{n}} =$$
$$a^{\frac{m}{n}}\left[1+\frac{m}{n}\left(\frac{x}{a+x}\right)+\frac{m}{n}.\frac{m+n}{2n}\left(\frac{x}{a+x}\right)^2+\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}\right.$$
$$\left.\times\left(\frac{x}{a+x}\right)^3+\text{\&c.}\right]$$

$$\text{Or } (a+x)^{\frac{m}{n}} =$$
$$2a^{\frac{m}{n}}\left[1-\frac{m}{n}\left(\frac{a-x}{a+x}\right)+\frac{m}{n}.\frac{m+n}{2n}\left(\frac{a-x}{a+x}\right)^2-\frac{m}{n}.\frac{m+n}{2n}.\frac{n+2n}{3n}\right.$$
$$\left.\times\left(\frac{a-x}{a+x}\right)^3+\text{\&c.}\right]$$

It may here also be observed, that if m be made to represent any whole or fractional number, whether positive or negative, the first of these expressions may be exhibited in the more simple form

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{1.2}a^{m-2}x^2 + \frac{m(m-1)}{1.2}$$
$$\frac{(m-2)}{3}a^{m-3}x^3\ldots.$$
$$\ldots\ldots\frac{m(m-1)(m-2)\ldots\ldots[m-(n-1)]a^nx^{m-n}}{1.2.3.4\ldots\ldots\ldots\ldots n}$$

Where the last term is called the *general term of the series*, because if 1, 2, 3, 4, &c., be substituted successively for n, it will give all the rest.

EXAMPLES.

1. It is required to convert $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

Here $P=a^2$, $Q=\frac{x}{a^2}$, $\frac{m}{n}=\frac{1}{2}$, or $m=1$, and $n=2$.

Whence

$$P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{\frac{1}{2}}=a=A,$$

$$\frac{m}{n}AQ=\frac{1}{2}\times\frac{a}{1}\times\frac{x}{a^2}=\frac{x}{2a}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{x}{2a}\times\frac{x}{a^2}=-\frac{x^2}{2.4a^3}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times-\frac{x^2}{2.4a^3}\times\frac{x}{a^2}=\frac{3x^3}{2.4.6a^5}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{3x^3}{2.4.6a^5}\times\frac{x}{a^2}=-\frac{3.5x^4}{2.4.6.8a^7}=E,$$

$$\frac{m-4n}{5n}EQ=\frac{1-8}{10}\times-\frac{3.5x^4}{2.4.6.8a^7}\times\frac{x}{a^2}=\frac{3.5.7x^5}{2.4.6.8.10a^9}=F,$$

&c. &c. &c.

Therefore $(a^2+x)^{\frac{1}{2}}=$

$$a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+\frac{3.5.7x^5}{2.4.6.8.10a^9}-\text{\&c.}$$

Where the law of formation of the several terms of the series is sufficiently evident.

2. It is required to convert $\frac{1}{(a+b)^2}$ or its equal $(a+b)^{-2}$, into an infinite series.

Here $P=a$, $Q=\frac{b}{a}$, and $\frac{m}{n}=-2$, or $m=-2$ and $n=1$,

whence

$$P^{\frac{m}{n}}=(a)^{\frac{m}{n}}=a^{-2}=\frac{1}{a^2}=A,$$

$$\frac{m}{n}\text{AQ}=-\frac{2}{1}\times\frac{1}{a^2}\times\frac{b}{a}=-\frac{2b}{a^3}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-2-1}{2}\times-\frac{2b}{a^3}\times\frac{b}{a}=\frac{3b^2}{a^4}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{-2-2}{3}\times\frac{3b^2}{a^4}\times\frac{b}{a}=-\frac{4b^3}{a^5}=\text{D},$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{-2-3}{4}\times-\frac{4b^3}{a^5}\times\frac{b}{a}=\frac{5b^4}{a^6}=\text{E},$$

&c. &c. &c.

Consequently $\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}$ &c.

3. It is required to convert $\frac{a^2}{(a^2-x)^{\frac{1}{2}}}$, or its equal $a^2(a^2-x)^{-\frac{1}{2}}$, into an infinite series.

Here

$\text{P}=a^2$, $\text{Q}=-\frac{x}{a^2}$, and $\frac{m}{n}=\frac{-1}{2}$, or $m=-1$ and $n=2$;

Whence

$$\text{P}^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{-\frac{1}{2}}=\frac{1}{a}=\text{A},$$

$$\frac{m}{n}\text{AQ}=-\frac{1}{2}\times\frac{1}{a}\times-\frac{x}{a^2}=\frac{x}{2a^3}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-1-2}{4}\times\frac{x}{2a^3}\times-\frac{x}{a^2}=\frac{3x^2}{2.4a^5}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{-1-4}{6}\times\frac{3x^2}{2.4a^5}\times-\frac{x}{a^2}=\frac{3.5x^3}{2.4.6a^7}=\text{D},$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{-1-6}{8}\times\frac{3.5x^3}{2.4.6a^7}\times-\frac{x}{a^2}=\frac{3.5.7x^4}{2.4.6.8a^9}=\text{E},$$

&c. &c. &c

Therefore

$$\frac{1}{(a^2-x)^{\frac{1}{2}}}=\frac{1}{a}+\frac{1}{2}\left(\frac{x}{a^3}\right)+\frac{3}{2.4}\left(\frac{x^2}{a^5}\right)+\frac{3.5}{2.4.6}\left(\frac{x^3}{a^7}\right)+\frac{3.5.7}{2.4.6.8}\left(\frac{x^4}{a^9}\right)$$
$+$ &c.

And

$$\frac{a^2}{(a^2-x)^{\frac{1}{2}}}=a+\frac{1}{2}\left(\frac{x}{a}\right)+\frac{3}{2.4}\left(\frac{x^2}{a^3}\right)+\frac{3.5}{2.4.6}\left(\frac{x^3}{a^5}\right)+\frac{3.5.7}{2.4.6.8}\left(\frac{x^4}{a^7}\right)$$
$$+\ \&c.$$

4. It is required to convert $\sqrt[3]{9}$, or its equal $(8+1)^{\frac{1}{3}}$, into an infinite series.

Here $P=8$, $Q=\frac{1}{8}$, and $\frac{m}{n}=\frac{1}{3}$, or $m=1$ and $n=3$.

Whence

$$P^{\frac{m}{n}}=(8)^{\frac{m}{n}}=8^{\frac{1}{3}}=2=A,$$

$$\frac{m}{n}AQ=\frac{1}{3}\times\frac{2}{1}\times\frac{1}{2^3}=\frac{1}{3.2^2}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{1}{3.2^2}\times\frac{1}{2^3}=-\frac{1}{3.6.2^4}=C$$

$$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times-\frac{1}{3.6.2^4}\times\frac{1}{2^3}=\frac{5}{3.6.9.2^7}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{1-9}{12}\times\frac{5}{3.6.9.2^7}\times\frac{1}{2^3}=-\frac{5.8}{3.6.9.12.2^{10}}=E,$$

$$\frac{m-4n}{5n}EQ=\frac{1-12}{15}\times-\frac{5.8}{3.6.9.12.2^{10}}\times\frac{1}{2^3}=\frac{5.8.11}{3.6.9.12.15.2^{13}},$$

&c. &c. &c.

Therefore $\sqrt[3]{9}=$

$$2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\frac{5.8.11}{3.6.9.12.15.2^{13}}$$
$$-\&c.$$

5. It is required to convert $\sqrt{2}$, or its equal $\sqrt{(1+1)}$, into an infinite series.

Ans. $1+\frac{1}{2}-\frac{1}{2.4}+\frac{1.3}{2.4.6}-\frac{1.3.5}{2.4.6.8}+\frac{1.3.5.7}{2.4.6.8.10}-$ &c.

6. It is required to convert $\sqrt[3]{7}$, or its equal $(8-1)^{\frac{1}{3}}$, into an infinite series.

Ans. $2-\frac{1}{3.2^2}-\frac{1}{3.6.2^4}-\frac{1.5}{3.6.9.2^7}-\frac{1.5.8}{3.6.9.12.2^{10}}-$ &c.

7. It is required to convert 240, or its equal $(243-3)^{\frac{1}{5}}$, into an infinite series.

$$Ans.\ 3-\frac{1}{5.3^3}-\frac{4}{5.10.3^7}-\frac{4.9}{5.10.15.3^{11}}-\frac{4.9.14}{5.10.15.20.3^{15}}-\&c.$$

8. It is required to convert $(a \pm x)^{\frac{1}{2}}$ into an infinite series.

$$Ans.\ a^{\frac{1}{2}}\{1 \pm \frac{x}{2a}-\frac{x^2}{2.4a^2} \pm \frac{3x^3}{2.4.6a^3}-\frac{3.5x^4}{2.4.6.8a^4} \pm \&c.\}$$

9. It is required to convert $(a \pm b)^{\frac{1}{3}}$ into an infinite series.

$$Ans.\ a^{\frac{1}{3}}\{1 \pm \frac{b}{3a}-\frac{2b^2}{3.6a^2} \pm \frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4} \pm \&c.\}$$

10. It is required to convert $(a-b)^{\frac{1}{4}}$ into an infinite series.

$$Ans.\ a^{\frac{1}{4}}\{1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-\&c.\}$$

11. It is required to convert $(a+x)^{\frac{2}{3}}$ into an infinite series.

$$Ans.\ a^{\frac{2}{3}}\{1+\frac{2x}{3a}-\frac{x^2}{9a^2}+\frac{4x^3}{9^2a^3}-\frac{4.7x^4}{9^2 12a^4}+\frac{4.7.10x^5}{9^2.12.15a^5}-\&c.\}$$

12. It is required to convert $(1-x)^{\frac{2}{5}}$ into an infinite series.

$$Ans.\ 1-\frac{2x}{5}-\frac{2.3x^2}{5.10}-\frac{2.3.8x^3}{5.10.15}-\frac{2.3.8.13x^4}{5.10.15.20}-\&c.$$

13. It is required to convert $\frac{1}{(a \pm x)^{\frac{1}{2}}}$, or its equal $(a+x)^{-\frac{1}{2}}$, into an infinite series.

$$Ans.\ \frac{1}{a^{\frac{1}{2}}}\{1 \mp \frac{x}{2a}+\frac{3x^2}{2.4a^2} \mp \frac{3.5x^3}{2.4.6a^3}+\frac{3.5.7x^4}{2.4.6.8a^4} \mp \&c.\}$$

14. It is required to convert $\frac{a}{(a \pm x)^{\frac{1}{3}}}$, or its equal $a(a \pm x)^{-\frac{1}{3}}$, into an infinite series.

$$Ans.\ a^{\frac{2}{3}}\{1 \mp \frac{x}{3a}+\frac{4x^2}{3.6a^2} \mp \frac{4.7x^3}{3.6.9a^3}+\frac{4.7.10x^4}{3.6.9.12a^4} \mp \&c.\}$$

15. It is required to convert $\frac{1}{(1+x)^{\frac{1}{5}}}$, or its equal $(1+x)^{-\frac{1}{5}}$, into an infinite series.

Ans. $1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-$ &c.

16. It is required to convert $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$, or its equal $(a+x)\,(a^2-x^2)^{-\frac{1}{2}}$, into an infinite series.

Ans. $1+\frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{2a^3}+\frac{3x^4}{8a^4}+\frac{3x^5}{8a^5}+\frac{5x^6}{16x^6}+\frac{5x^7}{16a^7}+$ &c.

OF THE

INDETERMINATE ANALYSIS.

In the common rules of algebra, such questions are usually proposed as require some certain or definite answer; in which case it is necessary that there should be as many independent equations, expressing their conditions, as there are unknown quantities to be determined; or otherwise the problem would not be limited.

But in other branches of the science questions frequently arise that involve a greater number of unknown quantities than there are equations to express them; in which instances they are called indeterminate or unlimited problems; being such as usually admit of an indefinite number of solutions; although, when the question is proposed in integers, and the answers are required only in whole positive numbers, they are, in some cases, confined within certain limits, and in others the problem may become impossible.

PROBLEM I.

To find the integral values of the unknown quantities x and y in the equation

$$ax-by=\pm c, \text{ or } ax+by=c.$$

Where a and b are supposed to be given whole numbers, which admit of no common divisor, except when it is also a divisor of c.

RULE.

1. Let wh denote a whole or integral number; and reduce the equation to the form

$$x = \frac{by \pm c}{a} wh, \text{ or } x = \frac{c - by}{a} wh.$$

2. Throw all whole numbers out of that of these two expressions, to which the question belongs, so that the numbers d and e, in the remaining parts, may be each less than a; then

$$\frac{dy \pm e}{a} = wh, \text{ or } \frac{e - dy}{a} = wh.$$

3. Take such a multiple of one of these last formulæ corresponding with that above mentioned, as will make the coefficient of y nearly equal to a, and throw the whole numbers out of it as before.

Or find the sum or difference of $\frac{ay}{a}$ and the expression above used, or any multiple of it that comes near $\frac{ay}{a}$, and the result, in either of these cases, will still be a whole number.

4. Proceed in the same manner with this last result; and so on, till the coefficient of y becomes equal to 1, and the remainder equal to some number r; then

$$\frac{y \pm r}{a} = wh. = p, \text{ and } y = ap \mp r,$$

Where p may be o, or any integral number whatever that makes y positive.

And as the value of y is now known, that of x may be found from the given equation, when the question is possible.*

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* This rule is founded on the obvious principle, that the sum, difference, or product of any two whole numbers, is a whole number; and that, if a number divides the whole of any other number and a part of it, it will also divide the remaining part.

NOTE. Any indeterminate equation of the form

$$ax-by=\pm c,$$

in which a and b are prime to each other, is always possible, and will admit of an infinite number of answers in whole numbers.

But if the proposed equation be of the form

$$ax+by=c,$$

the number of answers will always be limited; and in some cases the question is impossible; both of which circumstances may be readily discovered, from the mode of solution above given.*

EXAMPLES.

1. Given $19x-14y=11$, to find x and y in whole numbers.

Here $x=\frac{14y+11}{19}=wh.$, and also $\frac{19y}{19}=wh.$

Whence, by subtraction, $\frac{19y}{19}-\frac{14y+11}{19}=\frac{5y-11}{19}=wh.$

Also, $\frac{5y-11}{19}\times 4=\frac{20y-44}{19}=y-2+\frac{y-6}{19}=wh.$

And by rejecting $y-2$, which is a whole number,

$$\frac{y-6}{19}=wh.=p.$$

---

* That the coefficients a and b, when these two formulæ are possible, should have no common divisor, which is not, at the same time, a divisor of c, is evident; for if $a=md$, and $b=me$, we shall have $ax\pm by=mdx\pm mey=c$; and consequently $dx+ey=\frac{c}{m}$.

But d, e, x, and y, being supposed to be whole numbers, $\frac{c}{m}$ must also be a whole number, which it cannot be, except when m is a divisor of c.

Hence, if it were required to pay $100l.$ in guineas and moidores only the question would be impossible; since, in the equation $21x+27y=2000$, which represents the conditions of the problem, the coefficients, 21 and 27, are each divisible by 3, whilst the absolute term 2000 is not divisible by it. See Vol. II. of my *Treatise on Algebra*, for the method of resolving questions of this kind, by means of *Continued Fractions.*

Whence we have $y=19p+6$.

And $x=\frac{14y+11}{19}=\frac{14(19p+6)+11}{19}=\frac{266p+95}{19}=14p+5$.

Where if p be taken $=0$ we shall have $x=5$, and $y=6$, for their least values; the number of solutions being obviously indefinite.

2. Given $2x+3y=25$, to determine x and y in whole positive numbers.

$$\text{Here } x=\frac{25-3y}{2}=12-y+\frac{1-y}{2}.$$

Hence, since x must be a whole number, it follows that $\frac{1-y}{2}$ must also be a whole number.

$$\text{Let therefore } \frac{1-y}{2}=wh=p;$$

$$\text{Then } 1-y=2p, \text{ or } y=1-2p.$$

$$\text{And } x=\frac{25-3y}{2}=\frac{25-3(1-2p)}{2}=11+3p.$$

Where p may be any whole number whatever that will render the values of x and y in these two equations positive.

But it is evident, from the value of y, that p must be either 0 or negative; and from that of x, that it must be 0, -1, -2, or -3.

$$\text{Whence, if } p=0,\ p=-1,\ p=-2,\ p=-3,$$

$$\text{Then } \begin{cases} x=11, & x=8, \ x=5, \ x=2, \\ y=1, & y=3, \ y=5, \ y=7, \end{cases}$$

which are all the answers in whole positive numbers that the question admits of.

3. Given $3x=8y-16$, to find the values of x and y in whole numbers.

$$\text{Here } x=\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}=wh.; \text{ or } \frac{2y-1}{3}=wh.$$

$$\text{Also } \frac{2y-1}{3}\times 2=\frac{4y-2}{3}=y+\frac{y-2}{3}=wh.$$

Or, by rejecting y, which is a whole number, there will remain $\frac{y-2}{3}=wh.=p.$

$$\text{Therefore } y=3p+2,$$

$$\text{and } x=\frac{8y-16}{3}=\frac{8(3p+2)-16}{3}=\frac{24p}{3}=8p.$$

Where if p be put $=1$, we shall have $x=8$ and $y=5$, for their least values; the number of answers being, as in the first question, indefinite.

4. Given $21x+17y=2000$, to find all the possible values of x and y in whole numbers.

$$\text{Here } x=\frac{2000-17y}{21}=95+\frac{5-17y}{21}=wh.\,;$$

$$\text{or, omitting the 95, } \frac{5-17y}{21}=wh.\,;$$

$$\text{consequently, by addition, } \frac{21y}{21}+\frac{5-17y}{21}=\frac{4y+5}{21}=wh.$$

$$\text{Also, } \frac{4y+5}{21}\times5=\frac{20y+25}{21}=1+\frac{4+20y}{21}=wh.\,;$$

$$\text{or, by rejecting the whole number, } \frac{4+20y}{21}=wh.$$

$$\text{And, by subtraction, } \frac{21y}{21}-\frac{4+20y}{21}=\frac{y-4}{21}=wh., =p.$$

$$\text{Whence } y=21\,p+4,$$

$$\text{and } x=\frac{2000-17y}{21}=\frac{2000-17(21p+4)}{21}=92-17p.$$

Where, if p be put $=0$, we shall have the least value of $y=4$, and the corresponding, or greatest value of $x=92$.

And the rest of the answers will be found by adding 21 continually to the least value of y, and subtracting 17 from the greatest value of x; which being done, we shall obtain the six following results:

$x=92$	75	58	41	24	7
$y=4$	25	46	67	88	109

These being all the solutions in whole numbers that the question admits of.

Note 1. When there are three or more unknown quantities, and only one equation by which they can be determined, as

$$ax+by+cz=d,$$

it will be proper first to find the limit of the quantity that has the greatest coefficient, and then to ascertain the different values of the rest, by separate substitutions of the several values of the former, from 1 up to that extent, as in the following question.

5. Given $3x+5y+7z=100$, to find all the different values of x, y, and z, in whole numbers.*

Here each of the least integer values of x and y is 1; whence it follows, that

$$z=\frac{100-5-3}{7}=\frac{92}{7}=13\tfrac{1}{7}.$$

Consequently z cannot be greater than 13.

By proceeding, therefore, as in the former rule, we shall have

$$x=\frac{100-5y-7z}{3}=33-y-2z+\frac{1-2y-z}{3}=wh.;$$

and, by rejecting $33-y-2z$,

$$\frac{1-2y-z}{3}=wh.; \text{ or } \frac{3y}{3}+\frac{1-2y-z}{3}=\frac{y+1-z}{3}=wh.$$

$$\text{Whence } \frac{y+1-z}{3}=p,$$

$$\text{or } y=3p+z-1.$$

---

* If any indeterminate equation, of the kind above given, has one or more of its coefficients, as c negative, the equation may be put under the form

$$ax+by=d+cz,$$

in which case it is evident that an indefinite number of values may be given to the second side of the equation, by means of the indefinite quantity z; and consequently also to x and y, in the first.

And if the coefficients a, b, c, in any such equation, have a common divisor, while the absolute number d has not, the question, as in the first case, becomes impossible. For the reason of which, see vol. I. of my *Treatise on Algebra*, before quoted.

And, consequently, putting $p=0$, we shall have the least value of $y=z-1$; now, by taking $z=1$, y becomes $=0$, and $x=31$; but this answer is inadmissible, because $y=0$ is not an integer; but by adding 3, the coefficient of x, to this value of y, and subtracting 5, the coefficient of y, from the value of x, we shall obtain another answer. By repeating this process continually we shall obtain all the possible values of x and y for this value of z; and in a similar manner are the values of x and y to be found when $z=2$, &c., when all the possible solutions will be found to be 41 in number, as follows:

$z=1$	$y=$	3	6	9	12	15	18
	$x=$	26	21	16	11	6	1
$z=2$	$y=$	1	4	7	10	13	16
	$x=$	27	22	17	12	7	2
$z=3$	$y=$	2	5	8	11	14	
	$x=$	23	18	13	8	3	
$z=4$	$y=$	3	6	9	12		
	$x=$	19	14	9	4		
$z=5$	$y=$	1	4	7	10		
	$x=$	20	15	10	5		
$z=6$	$y=$	2	5	8	11		
	$x=$	16	11	6	1		
$z=7$	$y=$	3	6	9			
	$x=$	12	7	2			
$z=8$	$y=$	1	4	7			
	$x=$	13	8	3			
$z=9$	$y=$	2	5				
	$x=$	9	4				
$z=10$	$y=$	3					
	$x=$	5					
$z=11$	$y=$	1	4				
	$x=$	6	1				
$z=12$	$y=$	2					
	$x=$	2					

Note 2. If there be three unknown quantities, and only two equations for determining them, as

$$ax+by+cz=d, \text{ and } ex+fy+gz=h,$$

exterminate one of these quantities in the usual way, and find the values of the other two from the resulting equation, as before.

Then, if the values, thus found, be separately substituted in either of the given equations, the corresponding values of the remaining quantities will likewise be determined: Thus,

6. Let there be given $x-2y+z=5$, and $2x+y-z=7$, to find the values of x, y, and z.

Here, by multiplying the first of these equations by 2, and subtracting the second from the product, we shall have

$$3z-5y=3, \text{ or } z=\frac{3+5y}{3}=1+y+\frac{2y}{3}=wh.;$$

$$\text{and consequently } \frac{2y}{3}, \text{ or } \frac{3y}{3}-\frac{2y}{3}=\frac{y}{3}=wh.=p.$$

Whence $y=3p$.

And by taking $p=1$, 2, 3, 4, &c., we shall have $y=3$, 6, 9, 12, 15, &c., and $z=6$, 11, 16, 21, 26, &c.

But from the first of the two given equations,

$$x=5+2y-z;$$

whence, by substituting the above values for y and z, the results will give $x=5$, 6, 7, 8, 9, &c.

And therefore the first six values of x, y, and z, are as below:

$x=5$	6	7	8	9	10
$y=3$	6	9	12	15	18
$z=6$	11	16	21	26	31

Where the law by which they can be continued is sufficiently obvious.

EXAMPLES FOR PRACTICE.

1. Given $3x=8y-16$, to find the least values of x and y in whole numbers. *Ans.* $x=8$, $y=5$.

2. Given $14x=5y+7$, to find the least values of x and y in whole numbers. *Ans.* $x=3$, $y=7$.

3. Given $27x=1600-16y$, to find the least values of x and y in whole numbers. *Ans.* $x=48$, $y=19$.

4. It is required to divide 100 into two such parts, that one of them may be divisible by 7, and the other by 11. *Ans.* The only parts are 56 and 44.

5. Given $9x+13y=2000$, to find the greatest value of x and the least value of y in whole numbers.

Ans. $x=215$, $y=5$.

6. Given $11x+5y=254$, to find all the possible values of x and y in whole numbers.

Ans. $x=19, 14, 9, 4$; $y=9, 20, 31, 42$.

7. Given $17x+19y+21z=400$, to find all the answers in whole numbers which the question admits of.

Ans. 10 different answers.

8. Given $5x+7y+11z=224$, to find all the possible values of x, y, and z, in whole positive numbers.

Ans. The number of answers is 59.

9. It is required to find in how many different ways it is possible to pay 20*l.* in half-guineas and half-crowns, without using any other sort of coin?

Ans. 7 different ways.

10. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louis-d'ors; how must I contrive to acquit myself of the debt, the louis being valued at 17*s.* apiece, and the guineas at 21*s.*?

Ans. I must give him 13 guineas, and he must give me 16 louis.

11. How many gallons of British spirits, at 12*s.*, 15*s.*, and 18*s.* a gallon, must a rectifier of compounds take to make a mixture of 1000 gallons, that shall be worth 17*s.* a gallon?

Ans. $111\frac{1}{9}$ at 12*s.*, $111\frac{1}{9}$ at 15*s.*, and $777\frac{7}{9}$ at 18*s.*

PROBLEM II.

To find such a whole number, as, being divided by other given numbers, shall leave given remainders.

RULE.

1. Call the number that is to be determined x, the numbers by which it is to be divided a, b, c, &c., and the given remainders f, g, h, &c.

2. Subtract each of the remainders from x, and divide the differences by a, b, c, &c., and there will arise

$$\frac{x-f}{a}, \frac{x-g}{b}, \frac{x-h}{c}, \text{\&c.,} = \text{whole numbers.}$$

3. Put the first of these fractions $\frac{x-f}{a}=p$, and substitute the value of x, as found in terms of p, from this equation, in the place of x in the second fraction.

4. Find the least value of p in this second fraction, by the last problem, which put $=r$, and substitute the value of x, as found in terms of r, in the place of x in the third fraction.

Find, in like manner, the least value of r, in this third fraction, which put $=s$, and substitute the value of x, as found in terms of s, in the fourth fraction, as before.

Proceed in the same way with the next following fraction, and so on, to the last; when the value of x, thus determined, will give the whole number required.

EXAMPLES.

1. It is required to find the least whole number, which, being divided by 17, shall leave a remainder of 7, and, when divided by 26, shall leave a remainder of 13.

Let $x =$ the number required.

Then $\frac{x-7}{17}$ and $\frac{x-13}{26} =$ whole numbers.

And, putting $\frac{x-7}{17}=p$, we shall have $x=17p+7$.

Which value of x, being substituted in the second fraction, gives $\frac{17p+7-13}{26}=\frac{17p-6}{26}=wh.$

But it is obvious that $\frac{26p}{26}$ is also $= wh.$

And consequently $\frac{26p}{26}-\frac{17p-6}{26}=\frac{9p+6}{26}=wh.$

Or $\frac{9p+6}{26}\times 3=\frac{27p+18}{26}=p+\frac{p+18}{26}=wh.$

Where, by rejecting p, there remains $\frac{p+18}{26}=wh.=r.$

Therefore $p=26r-18$;

whence, if r be taken $=1$, we shall have $p=8$.

And consequently $x=17p+7=17\times 8+7=143$, the number sought.

2. It is required to find the least whole number, which, being divided by 11, 19, and 29, shall leave the remainders 3, 5, and 10, respectively.

Let $x=$ the number required.

Then $\frac{x-3}{11}$, $\frac{x-5}{19}$, and $\frac{x-10}{29}=$ whole numbers.

And putting $\frac{x-3}{11}=p$, we shall have $x=11p+3$.

Which value of x being substituted in the second fraction, gives $\frac{11p-2}{19}=wh.$

$$\text{Or } \frac{11p-2}{19}\times 2=\frac{22p-4}{19}=p+\frac{3p-4}{19}=wh.$$

And, by rejecting p, there will remain $\frac{3p-4}{19}=wh.$

Also by mult^n. $\frac{3p-4}{19}\times 6=\frac{18p-24}{19}=\frac{18p-5}{19}-1=wh.$

Or, by rejecting the 1, $\frac{18p-5}{19}=wh.$

But $\frac{19p}{19}$ is likewise $=wh.$

Whence $\frac{19p}{19}-\frac{18p-5}{19}=\frac{p+5}{19}=wh$, which put $=r$.

Then we shall have

$p=19r-5$, and $x=11\ (19r-5)+3=209r-52$.

And if this value be substituted for x in the third fraction, there will arise

$$\frac{209r-62}{29}=7r-2+\frac{6r-4}{29}=wh$$

Or, by neglecting $7r-2$, we shall have the remaining part of the expression $\frac{6r-4}{29}=wh.$

But, by multiplication,

$$\frac{6r-4}{29}\times 5=\frac{30r-20}{29}=r+\frac{r-20}{29}=wh.$$

Or, by rejecting r, there will remain $\frac{r-20}{29}=wh$, which put $=s$.

Then $r=29s+20$; where, by taking $s=0$, we shall have $r=20$.

And consequently

$$x=209r-52=209\times 20-52=4128,$$

which is the number required.

3. To find a number, which, being divided by 6, shall leave the remainder 2, and when divided by 13, shall leave the remainder 3. *Ans.* 68.

4. It is required to find a number, which, being divided by 7, shall leave 5 for a remainder, and if divided by 9, the remainder shall be 2. *Ans.* 47.

5. It is required to find the least whole number, which, being divided by 39, shall leave the remainder 16, and when divided by 56, the remainder shall be 27. *Ans.* 1147.

6. It is required to find the least whole number, which, being divided by 7, 8, and 9, respectively, shall leave the remainders 5, 7, and 8. *Ans.* 215.

7. It is required to find the least whole number, which, being divided by each of the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, shall leave no remainders. *Ans.* 2520.

8. A person receiving a box of oranges, observed, that when he told them out by 2, 3, 4, 5, and 6 at a time, he had none remaining; but when he told them out by 7 at a time, there remained 5; how many oranges were there in the box, the number being the least possible? *Ans.* 180.

OF THE

DIOPHANTINE ANALYSIS.

This branch of algebra, which is so called from its inventor DIOPHANTUS, a Greek mathematician of Alexandria in Egypt, who flourished in or about the fourth century after CHRIST, relates chiefly to the finding of square and cube numbers, or to the rendering certain compound expressions free from surds; the method of doing which is by making such substitutions for the unknown quantity as will reduce the resulting equation to a simple one, and then finding the value of that quantity in terms of the rest.*

These questions are so curious and abstruse, that nothing less than the most refined algebra, applied with

* That DIOPHANTUS was not the inventor of algebra, as has been generally imagined, is obvious; since his method of applying it is such as could only have been used in an advanced state of the science: besides which, he nowhere speaks of the fundamental rules and principles, as an inventor certainly would have done, but treats of it as an art already sufficiently known; and seems to intend, not so much to teach it, as to cultivate and improve it, by solving such questions as, before his time, had been thought too difficult to be surmounted.

It is highly probable, therefore, that algebra was known among the Greeks long before the time of DIOPHANTUS; but that the works of preceding writers had been destroyed by the ravages of time, or the depredations of war and barbarism.

His *Arithmetical Questions*, out of which a considerable part of these problems are collected, consisted originally of thirteen books: but the first six only are now extant; the best edition of which is that published at Paris, by BACHET, in the year 1670, with Notes by FERMAT: in which work the subject is so skilfully handled, that the moderns, notwithstanding their other improvements, have been able to do little more than explain and illustrate his method. Those who have succeeded best in this respect, are VIETA, KERSEY, DE BILLY, OZANAM, PRESTET, SAUNDERSON, FERMAT, and EULER; the last of whom in particular has amplified and illustrated the Diophantine Algebra in as clear and satisfactory a manner as the subject seems to admit of.

The reader, who may be desirous of further information on this interesting subject, will find a methodical abstract of the several methods made use of by these writers, with a variety of examples to illustrate them, in the first and second volumes of my *Treatise on Algebra*, before mentioned.

the greatest skill and judgment, can surmount the difficulties which attend them. And in this respect no one, perhaps, has ever excelled Diophantus, or discovered greater knowledge of the extent and resources of the analytic art.

When we consider his work with attention, we are at a loss which to admire most—his singular sagacity, and the peculiar artifices he employs in forming such positions as the nature of the problems required, or the more than ordinary subtilty of his reasoning upon them.

Every particular question puts us upon a new way of thinking, and furnishes a fresh vein of analytical treasure, which cannot but prove highly useful to the mind in conducting it through other difficulties of this kind, whenever they may occur, as well as in enabling it to encounter, more readily, those that may arise in subjects of a different nature.

The following method of resolving these questions will be found of considerable service; but no general rule can be given, that will suit all cases; and therefore the solution must often be left to the ingenuity and skill of the learner.

RULE.

1. Put for the root of the square or cube required, one or more letters, such that, when they are involved, either the given number, or the highest power of the unknown quantity, may vanish from the equation; then if the unknown quantity be only of one dimension, the problem may be solved by reducing the equation.

2. But if the unknown quantity be still a square, or a higher power, some other new letters must be assumed to denote the root; with which proceed as before; and so on, till the unknown quantity is only of one dimension; when, from this, all the rest may be determined

EXAMPLES.

1. To divide a given square number (100) into two such parts, that each of them may be a square number.*

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* If $x - 10$ had been made the side of the second square, in the following solution of this question, instead of $2x - 10$, the equation

Let x^2 be one of the parts; then $100-x^2$ will be the other part; which is also to be a square number.

Assume the side of this second square $=2x-10$, then will $100-x^2=(2x-10)^2=4x^2-40x+100$; and, consequently, by reduction, $x=8$, and $2x-10=6$.

Therefore 64 and 36 are the parts required.

Or the same may be done, generally, thus:

Let $a^2 =$ given square number, $x^2 =$ one of its parts, and $a^2 - x^2 =$ the other; which is also to be a square number.

Assume the side of this second square $=rx-a$, then will $a^2-x^2=(rx-a)^2=r^2x^2-2arx+a^2$;

and, by reduction, $x=\frac{2ar}{r^2+1}$, and $rx-a=\frac{2ar^2}{r^2+1}-a$

$=\frac{2ar^2}{r^2+1}-\frac{ar^2+a}{r^2+1}=\frac{ar^2-a}{r^2+1}=$ side of the second square.

Therefore $\left(\frac{2ar}{r^2+1}\right)^2$ and $\left(\frac{ar^2-a}{r^2+1}\right)^2$ are the parts required; where a and r may be any whole numbers, taken at pleasure, provided r be greater than 1.*

2. To divide a given number (13) consisting of two known square numbers (9 and 4) into two other square numbers.

For the side of the first square sought, put $rx-3$; and for the side of the second, $sx-2$; r being the greater number, and s the less.

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would have been $x^2-20x+100=100-x^2$; in which case x, the side of the first square, would have been found $=10$, and $x-10$, or the side of the second square, $=0$; for which reason the substitution $x-10$ was avoided; but $3x-10$, $4x-10$, or any other quantity of the same kind, would have succeeded as well as the former, though the results would have been less simple.

* To this we may add the following useful property.

If s and r be any two unequal numbers, of which s is the greater, it can then be readily shown, from the nature of the problem, that $2rs$, s^2-r^2, and s^2+r^2 will be the perpendicular, base, and hypothenuse of a right-angled triangle.

From which expressions two square numbers may be found, whose sum or difference shall be square numbers; for $(2rs)^2+(s^2-r^2)^2=(s^2+r^2)^2$, and $(s^2+r^2)^2-(2rs)^2=(s^2-r^2)^2$, or $(s^2+r^2)^2-(s^2-r^2)^2=(2rs)^2$; where s and r may be any numbers whatever.

Then will $(rx-3)^2+(sx-2)^2=(r^2x^2-6rx+9)+(s^2x^2-4sx+4)=(r^2+s^2)x^2-(6r+4s)x+13=13$, or $(r^2+s^2)x^2=(6r+4s)x$.

From which last equation we have $x=\dfrac{6r+4s}{r^2+s^2}$.

Whence $rx-3=\dfrac{6r^2+4rs}{r^2+s^2}-3=\dfrac{3r^2+4rs-3s^2}{r^2+s^2}=$ side of the first square sought.

And $sx-2=\dfrac{6rs+4s}{r^2+s^2}-2=\dfrac{6rs-2r^2+2s^2}{r^2+s^2}=$ side of the second.

So that if r be taken $=2$, and $s=1$, we shall have $\dfrac{3r^2+4rs-3s^2}{r^2+s^2}=\dfrac{17}{5}$, and $\dfrac{6rs-2r^2+2s^2}{r^2+s^2}=\dfrac{6}{5}$ for the sides of the squares, in numbers, as was required.

And if a^2+b^2 be put equal to the number to be divided, the general solution may be obtained in the same way.*

3. To find two square numbers, whose difference shall be equal to any given number.

Let the difference d be resolved into any two unequal factors a and b; a being the greater, and b the less.

Also put x for the side of the less square sought, and $x+b$ for the side of the greater.

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* This question is considered by DIOPHANTUS as a very important one, being made the foundation of many of his other problems: it may be observed, that in the solution of it, given above, the values of r and s may be taken at pleasure, provided the ratio of them be not the same as that of 3 (a) to 2 (b); the reason of which restriction is, that if r and s were so taken, the sides of the squares sought would come out the same as the sides of the known squares which compose the given number, and therefore the operation would be useless.

The excellent JOHN KERSEY, after amplifying and illustrating this problem in a variety of ways, concludes his chapter thus:—'For a demonstration of the reverse of this *rare* speculation, see ANDERSONUS, Theorem 2, of VIETA's *mysterious Doctrine of Angular Sections;* and likewise HERIGONIUS, at the latter end of the first *tome* of his *Cursus Mathematicus*.'

Then $(x+b)^2-x^2=x^2+2bx+b^2-x^2=2bx+b^2=d=ab$ by the question.

And if this be divided by b, we shall have $2x+b=a$.

Whence $x=\frac{a-b}{2}=$ the side of the least square sought,

and $x+b=\frac{a-b}{2}+b=\frac{a+b}{2}=$ side of the greater.

So that taking $d=60$, and $a\times b=30\times 2$, we shall have $\frac{30-2}{2}=14$, and $\frac{30+2}{2}=16$.

Whence $(14)^2=196$, and $(16)^2=256$, for the squares in numbers; and so for any difference or factors whatever.

4. To find two numbers such, that if either of them be added to the square of the other, the sum shall be a square number.

Let the numbers sought be x and y.

Then $x^2+y=\square$, and $y^2+x=\square$.

And if $r-x$ be assumed for the side of the first square, we shall have $x^2+y=r^2-2rx+x^2$, or cancelling x^2 on each side of the equation, $y=r^2-2rx$.

Therefore $2rx=r^2-y$, or $x=\frac{r^2-y}{2r}$.

Again, if $y+s$ be assumed for the side of the second square, we shall have $y^2+\frac{r^2-y}{2r}=(y+s)^2=y^2+2sy+s^2$.

Whence also $\frac{r^2-y}{2r}=2sy+s^2$, or $r^2-y=4rsy+2rs^2$.

And consequently, by transposition and division we shall have $y=\frac{r^2-2rs^2}{4rs+1}$ and $x=\frac{r^2-y}{2r}=\frac{2r^2s+s^2}{4rs+1}$.

So that $\frac{r^2-2rs^2}{4rs+1}$ and $\frac{2r^2s+s^2}{4rs+1}$ are the numbers required; where r and s may be taken at pleasure, provided r be greater than $2s^2$.

5. To find two numbers such, that their sum and difference shall be both square numbers.

Let x and x^2-x be the two numbers sought.

Then, since their sum is evidently a square number, one of the conditions of the question is fulfilled.

There remains, therefore, only their difference x^2-2x to be made a square.

And if, for the side of this square, there be put $x-r$, we shall have $x^2-2rx+r^2=x^2-2x$, or $2rx-2x=r^2$.

Whence $x=\frac{r^2}{2r-2}$ and $x^2-x=\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$.

So that $\frac{r^2}{2r-2}$ and $\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$ are the numbers required, where r may be taken at pleasure, provided it be greater than 1.

6. To find three numbers such, that not only the sum of all three of them, but also the sum of every two, shall be a square number.

Let $4x$, x^2-4x, and $2x+1$, be the three numbers sought.

Then $(4x)+(x^2-4x)=x^2$, $(x^2-4x)+(2x+1)=x^2-2x+1$, and $(4x+x^2-4x+2x+1)=x^2+2x+1$, are evidently squares.

And, therefore, three of the conditions mentioned in the question are fulfilled.

Whence it remains only to make the quantity $(4x)+(2x+1)$, or $6x+1=$ to a square.

Let, therefore, $6x+1=a^2$; and we shall have, by transposition and division, $x=\frac{a^2-1}{6}$.

And, consequently, $\frac{4a^2-4}{6}$, $\left(\frac{a^2-1}{6}\right)^2-\frac{4a^2-4}{6}$, and

$\frac{2a^2-2}{6}+1$; or $\frac{2a^2-2}{3}$, $\frac{a^4-26a^2+25}{36}$, and $\frac{a^2+2}{3}$

are the numbers required; where a may be any number taken at pleasure, provided it be greater than 5.

7. To find three square numbers, such that the sum of every two of them shall be a square number.*

Let x^2, y^2, and z^2, be the numbers sought; then $x^2+z^2=\square$, $y^2+z^2=\square$, and $x^2+y^2=\square$

or $\frac{x^2}{z^2}+1=\square$, $\frac{y^2}{z^2}+1=\square$, and $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\square$.

And, by putting $\frac{x}{z}=\frac{s^2-1}{2s}$, and $\frac{y}{z}=\frac{r^2-1}{2r}$, we shall have $\frac{x^2}{z^2}+1=\frac{s^4+2s^2+1}{4s^2}$, and $\frac{y^2}{z^2}+1=\frac{r^4+2r^2+1}{4r^2}$, which are both evidently squares; and therefore it only remains to make $\frac{x^2}{z^2}+\frac{y^2}{z^2}=$ square number.

But $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\left(\frac{s^2-1}{2s}\right)^2+\left(\frac{r^2-1}{2r}\right)^2=\frac{(s^2-1)^2}{4s^2}+\frac{(r^2-1)^2}{4r^2}=$ $\frac{r^2\times(s^2-1)^2+s^2\times(r^2-1)^2}{4r^2s^2}=$ square number.

Or, by rejecting $4r^2s^2$, $r^2\times(s^2-1)^2+s^2\times(r^2-1)^2=$ $r^2\times(s+1)^2\times(s-1)^2+s^2\times(r+1)^2\times(r-1)^2=$ to a square number.

And, therefore, by making $r-1=s+1$, or $r=s+2$, we shall have $(s+2)^2\times(s+1)^2\times(s-1)^2+s^2\times(s+3)^2\times(s+1)^2$ $=$ to a square number.

Or, $(s+2)^2\times(s-1)^2+s^2\times(s+3)^2=2s^4+8s^3+6s^2-4s+4$ $=$ to a square number.

Hence, in order to resolve this expression, let its root be assumed $=\frac{5}{4}s^2-s+2$.

Then, $2s^4+8s^3+6s^2-4s+4=(\frac{5}{4}s^2-s+2)^2=\frac{25}{16}s^4-\frac{5}{2}s^3+$ $6s^2-4s+4$; or $2s^4+8s^3=\frac{25}{16}s^4-\frac{5}{2}s^3$; or $2s+8=\frac{25}{16}s-\frac{5}{2}$.

From which we have $s=-24$, and $r=-22$.

* This question, like many others here proposed, is capable of a great variety of answers; but the least roots, which have yet been found, in whole numbers, are 44, 117, and 240. These were first given by Saunderson, in Vol. II. of his *Algebra*; and are to be found also in Euler's *Algebra*, English Translation, Vol. II., which is a work abounding with a great variety of particulars relating to the more abstruse parts of the Diophantine analysis.

$$\text{And } \frac{x}{z}=\frac{s^2-1}{2s}=-\frac{575}{48}, \text{ and } \frac{y}{z}=\frac{r^2-1}{2r}=-\frac{483}{44};$$

$$\text{or } x=-\frac{575z}{48} \text{ and } y=-\frac{483z}{44}.$$

Wherefore, to obtain the answer in whole numbers, let z be taken$=528$, and we shall have $x=-6325$, and $y=-5796$.

Consequently, 528, -5796, and -6325, are the roots of the squares, and 528^2, 5796^2, and 6325^2, are the squares required.

EXAMPLES FOR PRACTICE.

1. It is required to find a number x such, that $x+1$ and $x-1$ shall be both square numbers.* *Ans.* $x=\frac{5}{4}$.

2. It is required to find a number x such, that $x+128$ and $x+192$ shall be both squares. *Ans.* $x=97$.

3. It is required to find a number such, that x^2+x and x^2-x shall be both squares. *Ans.* $\frac{25}{24}$.

4. It is required to find two numbers such, that if each of them be added to their product, the sums shall be both squares. *Ans.* $\frac{2}{3}$ and $\frac{5}{3}$.

5. It is required to find three square numbers in arithmetical progression. *Ans.* 1, 25, and 49.

6. To find three whole numbers in arithmetical progression such, that the sum of every two of them shall be a square number. *Ans.* 482, 3362, and 6242.

7. To find three numbers, such, that if to the square of each the sum of the other two be added, the three sums shall be all squares. *Ans.* 1, $\frac{8}{3}$, and $\frac{16}{3}$.

8. To find two numbers in proportion as 8 is to 15, and such that the sum of their squares shall be a square number.

Ans. Any two numbers having the ratio of 8 to 15.

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* The answers to many of the questions here given cannot be found in whole numbers.

9. To find two numbers such, that if the square of each be added to their product, the sums shall be both squares. *Ans.* 9 and 16.

10. To find two whole numbers such, that the sum or difference of their squares, when diminished by unity, shall be a square. *Ans.* 8 and 9.

11. It is required to resolve 4225, which is the square of 65, into as many other integral squares as the question admits of.

Ans. 16^2+63^2, 56^2+33^2, 60^2+25^2, and 52^2+39^2.

12. To find three numbers in geometrical proportion, such that each of them, when increased by a given number (19), shall be square numbers.

Ans. 81, $\frac{5}{4}$ and $\frac{25}{1296}$.

13. To find two numbers, such that if their product be added to the sum of their squares, the result shall be a square number. *Ans.* 5 and 3, 8 and 7, 16 and 5, &c.

14. To find three whole numbers, such that, if to the square of each the product of the other two be added, the three sums shall be all squares.

Ans. 9, 73, and 328.

15. It is required to find three square numbers, such that their sum, when added to each of their three roots, shall be all square numbers.

Ans. $\frac{4418}{62920}$, $\frac{13254}{62920}$, and $\frac{19881}{62920}$ = roots required.

16. To find three numbers in geometrical progression, such that if the mean be added to each of the extremes, the sums, in both cases, shall be squares.

Ans. 5, 20, and 80.

17. To find two numbers, such that not only each of them, but also their sum and their difference when increased by unity, shall be all square numbers.

Ans. 1368 and 840.

18. To find three numbers, such that whether their sum be added to, or subtracted from, the square of each of them, the numbers thence arising shall be all squares.

Ans. $\frac{406}{96}$, $\frac{518}{96}$, and $\frac{721}{96}$.

19. It is required to find three square numbers, such that the sum of their squares shall also be a square number.
Ans. 9, 16, and $\frac{144}{25}$.

20. It is required to find three square numbers, such that the difference of every two of them shall be a square number. *Ans.* 485809, 34225, and 23409.

21. It is required to divide any given cube number (8) into three other cube numbers. *Ans.* 1, $\frac{64}{27}$, and $\frac{125}{27}$.

22. To find three square numbers, such that the difference between every two of them, and the third, shall be a square number. *Ans.* 149^2, 241^2, and 269^2.

23. To find three cube numbers, such that if from each of them a given number (1) be subtracted, the sum of the remainders shall be a square number.
Ans. $\frac{4913}{3375}$, $\frac{21952}{3375}$, and 8.

OF THE

SUMMATION AND INTERPOLATION OF INFINITE SERIES.

The doctrine of Infinite Series is a subject which has engaged the attention of the greatest mathematicians both of ancient and modern times; and, when taken in its whole extent, is, perhaps, one of the most abstruse and difficult branches of abstract mathematics.

To find the sum of a series, the number of the terms of which is inexhaustible or infinite, has been regarded by some as a paradox, or a thing impossible to be done; but this difficulty will be easily removed, by considering that every finite magnitude whatever is divisible *ad infinitum*, or consists of an indefinite number of parts, the aggregate, or sum, of which, is equal to the quantity first proposed.

A number actually infinite is, indeed, a plain contradiction to all our ideas; for any number that we can possibly conceive, or of which we have any notion, must always be determinate and finite; being such that a greater may still be assigned, and a greater after this; and so on, without a possibility of ever coming to an end of the increase or addition.

This inexhaustibility, therefore, in the nature of numbers, is all that we can distinctly comprehend by their infinity; for though we can easily conceive that a finite quantity may become greater and greater without end, yet we are not, by that means, enabled to form any notion of the *ultimatum*, or last magnitude, which is incapable of further augmentation.

Hence, we cannot apply to an infinite series the common notion of a sum, or of a collection of several particular numbers, which are joined and added together, one after another; as this supposes that each of the numbers, composing that sum, is known and determined. But as every series generally observes some regular law, and continually approaches towards a term, or limit, we can easily conceive it to be a whole of its own kind, and that it must have a certain real value, whether that value be determinable or not.

Thus, in many series, a number is assignable beyond which no number of its terms can ever reach, or, indeed, be ever perfectly equal to it; but yet may approach towards it, in such a manner as to differ from it by less than any quantity that can be named. So that we may justly call this the value or sum of the series; not as being a number found by the common method of addition, but such a limitation of the value of the series, taken in all its infinite capacity, that, if it were possible to add all the terms together, on after another the sum would be equal to that number.

In other series, on the contrary, the aggregate or value of the several terms, taken collectively, has no limitation; which state of it may be expressed, by saying, that the sum of the series is infinitely great; or, that it has no determinate or assignable value, but may be carried on to such a length that its sum shall exceed any given number whatever.

Thus, as an illustration of the first of these cases, it may be observed, that if r be the ratio, g the greatest term, and l the least of any decreasing geometric series, the sum, according to the common rule, will be $(rg-l)\div(r-1)$: and if we suppose the less extreme, l, to be diminished till it becomes $=0$, the sum of the whole series will be $rg\div(r-1)$: for it is demonstrable, that the sum of no assignable number of terms of the series can ever be

equal to that quotient; and yet no number less than it will ever be equal to the value of the series.

Whatever consequences, therefore, follow from the supposition of $rg \div (r-1)$ being the true and adequate value of the series, taken in all its infinite capacity, as if all the parts were actually determined, and added together, no assignable error can possibly arise from them, in any operation, or demonstration, where the sum is used in that sense; because, if it should be said that the series exceeds that value, it can be proved that this excess must be less than any assignable difference; which is in effect no difference at all; whence the supposed error cannot exist; and consequently $rg \div (r-1)$ may be looked upon as expressing the true value of the series, continued to infinity.

We are, also, further satisfied of the reasonableness of this doctrine, by finding, in fact, that a finite quantity is frequently convertible into an infinite series, as appears in the case of circulating decimals. Thus two thirds expressed decimally, is $\frac{2}{3} = .66666\,.. = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} +$ &c. continued *ad infinitum*. But this is a geometric series, the first term of which is $\frac{6}{10}$, and the ratio $\frac{1}{10}$; and therefore the sum of all its terms, continued to infinity, will evidently be equal to $\frac{2}{3}$, or the number from which it was originally derived. And the same may be shown of many other series, and of all circulating decimals in general.

With respect to the processes by which the summation of various kinds of infinite series are usually obtained, one of the principal is by the method of differences, pointed out and illustrated in Prob. IV. following.

Another method is that first employed by James and John Bernoulli, which consists in resolving the given series into others of a different kind, of which the summation can be determined; or by subtracting from an assumed series, the value of which is denoted by the same series, deprived of some of its first terms; in which case a new series will arise, whose sum will be known.

A third method, which is that of Demoivre, consists in putting the sum of the series $=$ s, and multiplying each side of the equation by some binomial or trinomial expression, which involves the powers of the unknown quantity x, and certain known coefficients, then taking x, after the

process is performed, of such a value that the assumed binomial, &c., shall become $= 0$, and transposing some of the first terms; when a series will arise, the sum of which will be known, as before.

Each of which methods, modified so as to render it more commodious in practice, together with several other artifices for the same purpose, will be found sufficiently elucidated in the miscellaneous questions succeeding the following problems.

PROBLEM I.

Any series being given to find its several orders of differences.

RULE.

1. Take the first term from the second, the second from the third, the third from the fourth, &c., and the remainders will form a new series, called the *first order of differences.*

2. Take the first term of this last series from the second, the second from the third, the third from the fourth, &c., and the remainders will form another new series, called the *second order of differences.*

3. Proceed in the same manner for the third, fourth, fifth, &c., orders of differences; and so on till they terminate, or are carried as far as may be thought necessary.*

EXAMPLES.

1. Required the several orders of differences of the series $1, 2^2, 3^2, 4^2, 5^2, 6^2$, &c.

Here	1, 4, 9, 16, 25, 36, &c.	given series.
	3, 5, 7, 9, 11, &c.	1st diff.
	2, 2, 2, 2, &c.	2d diff.
	0, 0, 0, &c.	3d diff.

* When the several terms of the series continually increase, the differences will be all positive; but when they decrease, the differences will be negative and positive alternately.

2. Required the several orders of differences of the series 1, 2^3, 3^3, 4^3, 5^3, 6^3, &c.

Here 1,	8,	27,	64,	125,	216,	&c. given series.
	7,	19,	37,	61,	91,	&c. 1st diff.
		12,	18,	24,	30,	&c. 2d diff.
			6,	6,	6,	&c. 3d diff.
				0,	0,	&c. 4th diff.

3. Required the several orders of differences of the series 1, 3, 6, 10, 15, 21, &c.

Ans. 1st, 2, 3, 4, 5, &c.; 2d, 1, 1, 1, &c.

4. Required the several orders of differences of the series 1, 6, 20, 50, 105, 196, &c.

Ans. 1st, 5, 14, 30, 55, 91, &c.; 2d, 9, 16, 25, 36, &c.; 3d, 7, 9, 11, &c.; 4th, 2, 2, &c.

5. Required the several orders of differences of the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, &c.

PROBLEM II.

Any series, a, b, c, d, e, &c., being given to find the first term of the nth order of differences.

Let δ stand for the first term of the nth differences.

Then will $a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)}{2}\frac{(n-2)}{3}d + \frac{n(n-1)}{2.}\frac{(n-2)}{3.}\frac{(n-3)}{4.}e$, &c., to $n+1$ terms $= \delta$, when n is an even number.

And $-a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{2\ .\ 3}d - \frac{n(n-1)}{2}\frac{(n-2)(n-3)}{3\ .\ 4}e$, &c., to $n+1$ terms $= \delta$, when n is an odd number.*

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* When the terms of the several orders of differences happen to be very great, it will be more convenient to take the logarithms of the quantities concerned, whose differences will be smaller; and, when the operation is finished, the quantity answering to the last logarithm may be easily found.

EXAMPLES.

1. Required the first term of the third order of differences of the series 1, 5, 15, 35, 70, &c.

Here a, b, c, d, e, &c., $=1, 5, 15, 35, 70$, &c., respectively, and $n=3$,

Whence $-a+nb-\frac{n(n-1)}{2}c+\frac{n(n-1)(n-2)}{2.3}d=-a+3b-3c+d=-1+15-45+35=4$, the first term required.

2. Required the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, &c.

Here a, b, c, d, e, &c., $=1, 8, 27, 64$ 125, &c., respectively, and $n=4$.

Whence $a-nb+\frac{n(n-1)}{2}c-\frac{n(n-1)(n-2)}{2.3}d+\frac{n(n-1)(n-2)(n-3)}{2.3.4}e=a-4b+6c-4d+e=1-32+162-256+125=0$; so that the first term of the fourth order is 0.

3. Required the first term of the eighth order of differences of the series 1, 3, 9, 27, 81, &c. *Ans.* 256.

4. Required the first term of the fifth order of differences of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, &c.*

Ans. $-\frac{1}{32}$

PROBLEM III.

To find the nth term of the series a, b, c, d, e, &c., when the differences of any order become at last equal to each other.

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* The labour, in questions of this kind, may be often abridged by putting ciphers for some of the terms at the beginning of the series; by which means several of the differences will be equal to 0, and the answer, on that account, obtained in fewer terms.

RULE.

Let d', d'', d''', d^{iv}, &c., be the first term of the several orders of differences, found as in the last problem.

Then will $a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1.2}d''+\frac{(n-1)}{1}\frac{(n-2)(n-3)}{2.3}d'''+\frac{(n-1)(n-2)(n-3)(n-4)}{1.2.3.4}d^{iv},+$ &c. $=n$th term required.

EXAMPLES.

1. It is required to find the twelfth term of the series 2, 6, 12, 20, 30, &c.

Here	2,	6,	12,	20,	30,	&c. given series.
		4,	6,	8,	10,	&c. 1st diff.
			2,	2,	2,	&c. 2nd diff.
				0,	0,	&c. 3rd diff.

Whence 4 and 2 are the first terms of the differences.

Let, therefore, $4=d'$, $2=d''$, and $n=12$.

Then $a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1.2}d''=2+11\,d'+55d'$ $=2+44+110=156$, the 12th term, or the answer required.

2. Required the twentieth term of the series 1, 3, 6, 10, 15, 21, &c.

Here	1,	3,	6,	10,	15,	21,	&c. given series.
		2,	3,	4,	5,	6,	&c. 1st diff.
			1,	1,	1,	1,	&c. 2nd diff.
				0,	0,	0,	&c. 3rd diff.

Where 2 and 1 are the first terms of the differences.

Let, therefore, $2=d'$, $1=d''$, and $n=20$.

Then $a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1\,.\,2}d''=1+19d'+171d'$ $=1+38+171=210$, the 20th term required.

3. Required the fifteenth term of the series 1, 4, 9, 16, 25, 36, &c. *Ans.* 225

4. Required the twentieth term of the series 1, 8, 27, 64, 125, &c. *Ans.* 8000.

5. It is required to find the thirtieth term of the series $1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \frac{1}{21}, \frac{1}{28}$, &c. *Ans.* $\frac{1}{465}$.

PROBLEM IV.*

To find the sum of n terms of the series a, b, c, d, e, &c., when the differences of any order become at last equal to each other.

RULE.

Let d', d'', d''', d^{iv}, &c., be the first terms of the several orders of differences.

Then will $na + \frac{n(n-1)}{1 \,.\, 2} d' + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} d'' + \frac{n(n-1)(n-2)(n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} d''' + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} d^{\text{iv}}$ + &c., = to the sum of n terms of the series.

EXAMPLES.

1. Required the sum of n terms of the series 1, 2, 3, 4, 5, 6, &c.

Here	1,	2,	3,	4,	5,	6,	&c. given series.
		1,	1,	1,	1,	1,	&c. 1st diff.
			0,	0,	0,	0,	&c. 2nd diff.

where 1 and 0 are the first terms of the differences.

Let, therefore, $a=1$, and $d'=1$,

* When the differences in this or the former rule are finally $= 0$, any term, or the sum of any number of the terms, may be accurately determined; but if the differences do not vanish, the result is only an approximation; which, however, may be often very usefully applied in resolving various questions that may occur in this branch of the subject, and which will become continually nearer the truth as the differences diminish.

then will $na+n.\frac{n-1}{2}d'=n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=\frac{n(n+1)}{2}$ — the sum of n terms, as required.

2. Required the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, 5^2, &c., or 1, 4, 9, 16, 25, &c.

Here 1,	4,	9,	16,	25,	&c.	given series.
	3,	5,	7,	9,	&c.	1st diff.
		2,	2,	2,	&c.	2nd diff.
			0,	0,	&c.	3rd diff.

where 3 and 2 are the first terms of the differences.

Let, therefore, $a=1$, $d'=3$, $d''=2$.

Then will $na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\;.\;3}d''=$

$$n+\frac{3n(n-1)}{2}+\frac{2n(n-1)(n-2)}{2\;.\;3}=n+\frac{3n^2-3n}{2}+$$

$$\frac{n^3-3n^2+2n}{3}=\frac{2n^3+3n^2+n}{6}=\frac{n\times(n+1)\times(2n+1)}{6}=Ans.$$

3. Required the sum of n terms of the series 1^3, 2^3, 3^3, 4^3, 5^3, &c., or 1, 8, 27, 64, 125, &c.

Here 1,	8,	27,	64,	125,	&c.	given series.
	7,	19,	37,	61,	&c.	1st diff.
		12,	18,	24,	&c.	2nd diff.
			6,	6,	&c.	3rd diff.
				0,	&c.	4th diff.

where the first terms of the differences are 7, 12, and 6.

Let, therefore, $a=1$, $d'=7$, $d''=12$, and $d'''=6$.

Then will $na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\;.\;3}d''+\frac{n(n-1)}{2}$

$$\frac{(n-2)(n-3)}{3\;.\;4}d'''=n+\frac{7n(n-1)}{2}+\frac{12n(n-1)(n-2)}{2\;.\;3}+$$

$$\frac{6n(n-1)(n-2)(n-3)}{2\;.\;3\;.\;4}=n+\frac{7n^2-7n}{2}+2n^3-6n^2+4n+$$

$$\frac{n^4-6n^3+11n^2-6n}{4}=\frac{4n}{4}+\frac{14n^2-14n}{4}+\frac{8n^3-24n^2+16n}{4}$$

$$+\frac{n^4-6n^3+11n^2-6n}{4}=\frac{n^4+2n^3+n^2}{4}=\frac{n^2(n+1)^2}{4},$$ the sum of n terms as required.

4. Required the sum of n terms of the series 2, 6, 12, 20, 30, &c. *Ans.* $\dfrac{n\times(n+1)\times(n+2)}{3}$.

5. Required the sum of n terms of the series 1, 3, 6, 10, 15, &c. *Ans.* $\dfrac{n}{1}\times\dfrac{n+1}{2}\times\dfrac{n+2}{3}$.

6. Required the sum of n terms of the series 1, 4, 10 20, 35, &c. *Ans.* $\dfrac{n}{1}\times\dfrac{n+1}{2}\times\dfrac{n+2}{3}\times\dfrac{n+3}{4}$.

7. Required the sum of n terms of the series 1^4, 2^4, 3^4 4^4, &c., or 1, 16, 81, 256, &c.

Ans. $\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30}$.

8. Required the sum of n terms of the series 1^5, 2^5, 3^5 4^5, 5^5, &c. *Ans.* $\dfrac{n^6}{6}+\dfrac{n^5}{2}+\dfrac{5n^4}{12}-\dfrac{n^2}{12}$.

PROBLEM V.

Any series a, b, c, d, e, &c., of equidistant terms, being given, to find any intermediate term by interpolation.

RULE.

Let x be the place in the series, of any term z, that is to be interpolated; and d', d'', d''', &c., the first terms of the several orders of differences.

Then will $z = a+(x-1)d'+\dfrac{(x-1)(x-2)}{2}d''+$

$$\frac{(x-1)(x-2)(x-3)}{2\ .\ 3}d'''+\frac{(x-1)(x-2)(x-3)(x-4)}{2\ .\ 3\ .\ 4}d^{\text{iv}}+\text{ \&c.}$$

EXAMPLES.

1. Given the logarithmic sines of 1° 0′, 1° 1′, 1° 2′, and 1° 3 to find the log. sine of 1° 1′ 40″.

	1° 0′	1° 1′	1° 2′	1° 3′
Sines	8·2418553	8·2490332	8·2560943	8·2630424
	1st diff.	71779	70611	69481
		2d diff.	−1168	−1130
			3d diff.	38

Therefore 71779, −1168, and 38, are the 1st terms of the differences.

And since 1° 1′ 40″ falls between the second and third terms, and 1′ 40″ $=1\frac{2}{3}$, x will be $=1+1\frac{2}{3}=2\frac{2}{3}=\frac{8}{3}$, $d'=71779$, $d''=-1168$, and $d'''=38$.

Hence $z=a+(x-1)d'+\frac{(x-1)(x-2)}{2}d''+\frac{(x-1)}{1}\frac{(x-2)(x-3)}{2\,.\,3}=a+\frac{5}{3}d'+\frac{5}{9}d''-\frac{5}{81}d'''=8\cdot2418553+$ $\cdot0119632-\cdot0000649-\cdot0000002=8\cdot2537534$, the sine of 1° 1′ 40″, as was required.

2. Given the series $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$, &c., to find the term which stands in the middle between the two terms $\frac{1}{52}$ and $\frac{1}{53}$. *Ans.* $\frac{1}{52\cdot5}$.

3. Given the natural tangents of 88° 54′, 88° 55′, 88° 56′, 88° 57′, 88° 58′, 88° 59′, and 89°, to find the tangent of 88° 58′ 18″. *Ans.* 55·711145.

PROBLEM VI.

When the differences of any order of the series a, b, c, d, e, &c., are very small, or become equal to 0, any intermediate term may be interpolated as follows.

RULE.

Find the value of the unknown quantity in the equation

which stands against the given number of terms in the following table, and it will give the term required.*

1. $a-b=0$
2. $a-2b+c=0$
3. $a-3b+3c-d=0$
4. $a-4b+6c-4d+e=0$
5. $a-5b+10c-10d+5e-f=0$
6. $a-6b+15c-20d+15e-6f+g=0$
&c. &c.

Or universally for n terms.

$$a-nb+\frac{n(n-1)}{2}c-\frac{n(n-1)(n-2)}{2\,.\,3}d+\frac{n(n-1)(n-2)}{2\,.\,3}\,\frac{(n-3)}{4}e-\&c.=0.$$

EXAMPLES.

1. Given the logarithms of 101, 102, 104, and 105, to find the logarithm of 103.

Here the number of terms are 4.

And against 4 in the table, we have $a-4b+6c-4d+e=0$; or $c=\frac{4\times(b+d)-(a+e)}{6}=$ value of the unknown quantity, or term to be found,

Where, taking the logs. of 101, 102, 104, and 105, we have

$$a=2\cdot0043214$$
$$b=2\cdot0086002$$
$$d=2\cdot0170333$$
$$e=2\cdot0211893$$

And consequently

$$4\times(b+d)=16\cdot1025340$$
$$a+e=\ \ 4\cdot0255107$$
$$6)12\cdot0770233$$
$$2\cdot0128372=\text{log. of } 103,$$

as required.

* The more terms are given, in any series of this kind, the more accurately will the equation that is to be used approximate towards the true result, or answer required.

2. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50. *Ans.* 3·684032.

3. Given the logarithms of 50, 51, 52, 54, 55, and 56, to find the logarithm of 53. *Ans.* 1·72427586.

PROMISCUOUS EXAMPLES RELATING TO SERIES.

1. To find the sum (s) of n terms of the series 1, 2, 3, 4, 5, 6, &c.

$$\text{Let } 1+2+3+4+5+ \&\text{c.} \quad . \quad . \quad . \quad . \quad +n=\text{s}.$$
$$\text{Then } n+(n-1)+(n-2)+(n-3)+(n-4)$$
$$+(n-5)+\&\text{c.} \quad . \quad . \quad . \quad +1=\text{s}.$$

Whence, by addition,

$(n+1)+(n+1)+(n+1)+(n+1)+(n+1)+$ &c. . . . to n terms $=$ 2s.

And consequently $n(n+1)=2\text{s}$; or $\text{s}=\dfrac{n(n+1)}{2}$, the sum required.

2. To find the sum (s) of n terms of the series 1, 3, 5, 7, 9, 11, &c.

$$\text{Let } 1+3+5+7+9+\&\text{c.} \quad . \quad . \quad (2n-1)=\text{s}.$$
$$\text{Then } (2n-1)+(2n-3)+(2n-5)+ \quad . \quad . \quad . \quad +1=\text{s}$$

Whence, by addition,

$$2n+2n+2n+2n+2n+ \&\text{c. to } n \text{ terms} = 2\text{s},$$

and consequently $2n\times n=2\text{s}$

$$\text{or s} = \frac{2n^2}{2}=n^2 = \text{sum required.}$$

3. Required the sum (s) of n terms of the series $a+(a+d)+(a+2d)+(a+3d)+(a+4d)+$ &c.

Let $\text{s}=a+(a+d)+(a+2d)+(a+3d) \quad . \quad . \quad +\{a+(n-1)d\}$.

Then, by reversing the series,

$\text{s}=\{a+(n-1)d\}+\{a+(n-2)d\}+\{a+(n-3)d\} \; . \; +a.$

Whence, by addition, $2s=\{2a+(nd-d)\}+\{2a+(nd-d)\}+\{2a+(nd-d)\}+$ &c., to n terms,

and consequently $2s=(2a+nd-d)\times n$;

$$\text{or } s=\{2a+(n-1)d\}\times\frac{n}{2}, \text{ the sum required.}$$

Or the same may be determined in a different manner, as follows:

$$a+(a+d)+(a+2d)+(a+3d)+(a+4d) \text{ \&c.}$$
$$=\left|\begin{array}{l}(+1+1+1+1+1+\text{ \&c.})\times a\\(+0+1+2+3+4+\text{ \&c.})\times d\end{array}\right|=s.$$

But n terms of $1+1+1+1+1+$ &c. $=n$.

And n terms of $0+1+2+3+4$ &c. $=\dfrac{n(n-1)}{2}$.

$$\text{Whence } s=na+\frac{n(n-1)d}{2}=\{2a+(n-1)d\}\times\frac{n}{2},$$

which is the same answer as before.

4. To find the sum (s) of n terms of the series $1, x, x^2, x^3, x^4, x^5, x^6$, &c.

Let $1+x+x^2+x^3+x^4$, &c. . . . . $x^{n-1}=s$.

Then $x+x^2+x^3+x^4+x^5$, &c. . . . . $x^n=sx$.

Whence, by subtraction, $x^n-1=sx-s$.

$$\text{Or } s=\frac{x^n-1}{x-1}=\text{sum required.}$$

And when x is a proper fraction, the sum of the series, continued *ad infinitum*, may be found in the same manner.

Thus, putting $1+x+x^2+x^3+x^4+x^5$, &c. $=s$,

We shall have $x+x^2+x^3+x^4+x^5$, &c. $=sx$

And consequently $-1=sx-s$; or $s-sx=1$.

Whence $s=\dfrac{1}{1-x}=$ sum of an infinite number of terms of the series, as was to be found.

5. Required the sum (s) of the circulating decimal $\cdot 999999\ldots$ continued *ad infinitum*.*

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* Otherwise let

$$\cdot 999\ldots\text{ \&c.}=s.$$
$$\therefore\ 99\cdot 999\ldots\text{ \&c.}=100s.$$

Hence $99=99s$, or $s=1$.

Here $\cdot 999999\ldots=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}+$ &c.

$$=9\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\text{ \&c.}\right)=\text{s}$$

$$\text{Or } \frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\text{ \&c. }=\frac{\text{s}}{9}.$$

$$\text{Whence } 1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\text{ \&c.}=\frac{10\text{s}}{9}.$$

And consequently, by subtraction, $1=\frac{10\text{s}}{9}-\frac{\text{s}}{9}=\frac{9\text{s}}{9}=\text{s}$;

or s$=1$, the sum of the series.

6. Required the sum (s) of the series $a^2+(a+d)^2+(a+2d)^2+(a+3d)^2+(a+4d)^2+$ &c., continued to n terms.

Here

$$\begin{aligned}
&a^2=a^2\\
&(a+d)^2=a^2+2\times ad+d^2\\
&(a+2d)^2=a^2+2\times 2ad+4d^2\\
&(a+3d)^2=a^2+2\times 3ad+9d^2\\
&(a+4d)^2=a^2+2\times 4ad+16d^2\\
&\quad\text{\&c.}\qquad\qquad\qquad\text{\&c.}
\end{aligned}$$

Whence

$$\text{s}=\left|\begin{array}{l}\text{Sum of } n \text{ terms of } (1+1+1+1+1+\text{\&c.}\\ \quad+\ \ldots\ \text{ditto of } (0+1+2+3+4+\text{\&c.})2ad\\ \quad+\ \ldots\ \text{ditto of } (0+1+4+9+16+\text{\&c.})d^2\end{array}\right.$$

But n terms of $1+1+1+1+1+$&c.$=n$.

And of $0+1+2+3+4$ &c.$=\frac{n(n-1)}{1.2}$

Also of $0+1+4+9+$ &c.$=\frac{n(n-1)(2n-1)}{1.2.3}$

Therefore $\text{s}=na^2+n(n-1)ad+\frac{n(n-1)(2n-1)}{1.2.3}d^2$, the whole sum of the series to n terms.

7. Required the sum (s) of the series $a^3+(a+d)^3+(a+2d)^3+(a+3d)^3+(a+4d)^3+$ &c., continued to n terms.

Here, $a^3 = a^3$

$$\begin{aligned}
(a+d)^3 &= a^3 + 3\times a^2d + 3\times ad^2 + d^3\\
(a+2d)^3 &= a^3 + 3\times 2a^2d + 3\times 4ad^2 + 8d^3\\
(a+3d)^3 &= a^3 + 3\times 3a^2d + 3\times 9ad^2 + 27d^3\\
(a+4d)^3 &= a^3 + 3\times 4a^2d + 3\times 16ad^2 + 64d^3\\
(a+5d)^3 &= a^3 + 3\times 5a^2d + 3\times 25ad^2 + 125d^3\\
\&c. & \qquad \&c.
\end{aligned}$$

Whence

$$s = \left| \begin{array}{l} \text{Sum of } n \text{ terms of } (1+1+1+1+1 \ \&c.)a^3 \\ \quad + \ldots \text{ ditto of } (0+1+2+3+4+ \&c.)3a^2d \\ \quad + \ldots \text{ ditto of } (0+1+4+9+16+ \&c.)3ad^2 \\ \quad + \ldots \text{ ditto of } (0+1+8+27+64+ \&c.)d^3 \end{array} \right.$$

But n terms of $1+1+1+1+1+1$ &c. $= n$.

Ditto . . . of $0+1+2+3+4$ &c. $= \dfrac{n(n-1)}{1.2}$

Ditto . . . of $0+1+4+9+16$ &c.$= \dfrac{n(n-1)(2n-1)}{1.2.3}$

Ditto . . . of $0+1+8+27+64+$ &c. $= \dfrac{n^4-2n^3+n^2}{4}$

Therefore $s = na^3 + \dfrac{n(n-1)3a^2d}{2} + \dfrac{n(n-1)(2n-1)3ad^2}{6} +$ $\dfrac{(n^4-2n^3+n^2)d^3}{4}$, the sum of n terms, as was to be found.

8. Required the sum (s) of n terms of the series $1+3+7+15+31+$ &c.

Here the terms of this series are evidently equal to 1 $(1+2)$, $(1+2+4)$, $(1+2+4+8)$, &c., or to the successive sums of the geometrical series 1, 2, 4, 8, 16, &c.

Let, therefore, $a=1$, and $r=2$, and we shall have $a+ar+ar^2+ar^3+ar^4$, &c.$=1+2+4+8+16$. &c

But the successive sums of 1, 2, 3, 4, &c., terms of this series are,

1. $\dfrac{ar-a}{r-1} = (r-1) \times \dfrac{a}{r-1}$

2. $\dfrac{ar^2-a}{r-1} = (r^2-1) \times \dfrac{a}{r-1}$

$$3.\quad \frac{ar^3-a}{r-1}=(r^3-1)\times\frac{a}{r-1}$$

$$4.\quad \frac{ar^4-a}{r-1}=(r^4-1)\times\frac{a}{r-1}$$

&c. &c.

$$\text{Whence } s=\frac{a}{r-1}\times\left|\begin{array}{l} n \text{ terms of } r+r^2+r^3+r^4+\text{ \&c.}\\ -n \text{ terms of } 1+1+1+1+\text{ \&c.}\end{array}\right.$$

But $r+r^2+r^3+r^4+$ &c. $=(r^n-1)\times\dfrac{r}{r-1}$

And $1+1+1+1+$ &c. $=n$

Therefore $s=\dfrac{r(r^n-1)}{r-1}\times\dfrac{a}{r-1}-n\times\dfrac{a}{r-1}=2(2^n-1)-n$, the sum required.

9. It is required to find the sum of n terms of the series $\dfrac{1}{1}+\dfrac{3}{2}+\dfrac{7}{4}+\dfrac{15}{8}+\dfrac{31}{16}+\dfrac{63}{32}+$ &c.

Here the terms of this series are the successive sums of the geometrical series $\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}+\dfrac{1}{32}+$ &c.

Let, therefore, $a=1$ and $r=2$, then will

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{\&c.}=a+\frac{a}{r}+\frac{a}{r^2}+\frac{a}{r^3}+\frac{a}{r^4}+\frac{a}{r^5}+\frac{a}{r^6}\text{ \&c.}$$

But the successive sums of 1, 2, 3, 4, &c., terms of this series, are,

$$1.\quad \frac{(r-1)\times a}{(r-1)\times 1}=(r-1)\times\frac{a}{r-1}$$

$$2\quad \frac{(r^2-1)\times a}{(r-1)\times r}=\left(r-\frac{1}{r}\right)\times\frac{a}{r-1}$$

$$3.\quad \frac{(r^3-1)\times a}{(r-1)\times r^2}=\left(r-\frac{1}{r^2}\right)\times\frac{a}{r-1}$$

$$4.\quad \frac{(r^4-1)\times a}{(r-1)\times r^3}=\left(r-\frac{1}{r^3}\right)\times\frac{a}{r-1}$$

&c. &c.

Therefore

$$s = \frac{a}{r-1} \times \left| \begin{array}{l} n \text{ terms of } r+r+r+r+r+ \text{ \&c.} \\ -n \text{ terms of } \frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+ \text{ \&c.} \end{array} \right.$$

But $r+r+r+r+r+r+r+$ &c. $= nr$.

And $\frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+$ &c. $= \frac{r^n-1}{(r-1)r^{n-1}}$.

Whence

$s = \frac{a}{r-1} \times \left(nr - \frac{r^n-1}{(r-1)r^{n-1}}\right) = \frac{(n-1)2^n+1}{2^{n-1}}$, the required.

10. Required the sum (s) of the infinite series of the reciprocals of the triangular numbers $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$ &c continued to infinity.

Let $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c., *ad infinitum*, $=$ s.

Or $\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{2.3}+\frac{1}{2.5}+$ &c. . . . . $=$ s.

Then $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$ &c. . . . . $= \frac{s}{2}$.

That is,

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\text{\&c.} = \frac{s}{2}$$

$$\text{Or, } \left| \begin{array}{l} \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+ \text{ \&c.} \\ -\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+ \text{ \&c.} \end{array} \right| = \frac{s}{2}.$$

Whence $\frac{s}{2} = \frac{1}{1}$; or s$=2=$ sum required.

11. It is required to find the sum of n terms of the same series, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+$ &c.

Let $z = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. to $\frac{1}{n}$.

Then $z - \frac{1}{1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. to $\frac{1}{n}$.

And $z - \frac{1}{1} + \frac{1}{n+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. to $\frac{1}{n+1}$.

Therefore, by subtracting this from the first, we have

$\frac{1}{1} - \frac{1}{n+1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}$ &c. to $\frac{1}{n(n+1)}$.

Or $\frac{n}{n+1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}$ &c. to $\frac{1}{n(n+1)}$.

Whence $\frac{2n}{n+1} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} +$ &c. to $\frac{2}{n(n+1)}$.

Or $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} +$ &c. to $\frac{2}{n(n+1)} = \frac{2n}{n+1}$ the sum of n terms of the series, as was required.

12. It is required to find the sum of the series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6} +$ &c. continued to infinity.

Let $z = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ + &c. *ad infinitum.*

Then $z - \frac{1}{1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. by transposition.

And $1 = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}$ + &c. by subtraction

Or $1 - \frac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} +$ &c. by transposition.

Whence $\frac{1}{2} = \frac{4}{1.4.3} + \frac{6}{2.9.4} + \frac{8}{3.16.5} +$ &c. by subtraction,

or $\frac{1}{2} = \frac{2}{1.2.3} + \frac{2}{2.3.4} + \frac{2}{3.4.5} + \frac{2}{4.5.6}$ + &c.

$$\text{And } \frac{1}{2} \div 2 = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \text{\&c.}$$

But $\frac{1}{2} \div 2 = \frac{1}{4}$; therefore $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6} +$ *ad infinitum* $= \frac{1}{4}$, which is the sum required.

13. It is required to find the sum of n terms of the same series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6} +$ &c.

$$\text{Let } z = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \text{\&c., to } \frac{1}{n(n+1)}.$$

$$\text{Then } z - \frac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \text{\&c., to } \frac{1}{n(n+1)}.$$

And $z - \frac{1}{2} + \frac{1}{(n+1)\,(n+2)} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} +$ &c., continued to $\frac{1}{(n+1)\,(n+2)}$ terms.

Therefore $\frac{1}{2} - \frac{1}{(n+1)\,(n+2)} = \frac{2}{1.2.3} + \frac{2}{2.3.4} + \frac{2}{3.4.5} +$ &c., to n terms, by subtraction.

Whence $\frac{1}{4} - \frac{1}{2(n+1)\,(n+2)} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} +$ &c., to n terms, by division.

And consequently $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} +$ &c., continued to n terms $= \frac{1}{4} - \frac{1}{2(n+1)\,(n+2)}$, the sum required.

14. Required the sum (s) of the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} -$ &c., continued *ad infinitum*.

$$\text{Let } x = \frac{1}{2}, \text{ and s} = \frac{z}{1+x}.$$

$$\text{Then } \frac{z}{1+x} = (x - x^2 + x^3 - x^4 + x^5 \text{ \&c.})$$

$$\text{And } z = (1+x) \times (x - x^2 + x^3 - x^4 + x^5 \text{ \&c.})$$

Whence, by multiplication,

$$\begin{array}{l} x - x^2 + x^3 - x^4 + x^5 \text{ \&c.} \\ 1 + x \\ \hline x - x^2 + x^3 - x^4 + x^5 - \text{ \&c.} \\ \quad + x^2 - x^3 + x^4 - x^5 + \text{ \&c.} \\ \hline \end{array}$$

The sum of which is $= x + 0 + 0 + 0 + 0 +$ &c.

Therefore $z = x$, and $x - x^2 + x^3 - x^4 + x^5$ &c. $= \dfrac{x}{1+x}$.

Or $\dfrac{1}{2} - \dfrac{1}{4} + \dfrac{1}{8} - \dfrac{1}{16} + \dfrac{1}{32}$ &c. $= \dfrac{\frac{1}{2}}{1+\frac{1}{2}} = \dfrac{1}{3}$ the sum required.

15. Required the sum (s) of the series $\dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \dfrac{4}{16} +$ &c., continued *ad infinitum*.

$$\text{Let } x = \frac{1}{2} \text{ and s} = \frac{z}{(1-x)^2}.$$

$$\text{Then } \frac{z}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \text{ \&c.}$$

$$\text{And } z = (1-x)^2 \times (x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \text{ \&c.})$$

Whence, by multiplication,

$$\begin{array}{l} x + 2x^2 + 3x^3 + 4x^4 + \text{ \&c.} \\ 1 - 2x + x^2 \\ \hline x + 2x^2 + 3x^3 + 4x^4 \text{ \&c.} \\ \quad - 2x^2 - 4x^3 - 6x^4 \text{ \&c.} \\ \qquad\quad + \ x^3 + 2x^4 \text{ \&c.} \\ \hline \end{array}$$

The sum of which is $= x + 0 + 0 + 0 +$ &c.

Therefore $z = x$,

and $x+2x^2+3x^3+4x^4+5x^5$ &c. $=\dfrac{x}{(1-x)^2}$.

Or $\dfrac{1}{2}+\dfrac{2}{4}+\dfrac{3}{8}+\dfrac{4}{16}+\dfrac{5}{32}+\dfrac{6}{64}+$ &c. $=\dfrac{\frac{1}{2}}{(1-\frac{1}{2})^2}=2$,

the sum of the infinite series required.

16. It is required to find the sum (s) of the series $\dfrac{1}{3}+\dfrac{4}{9}+\dfrac{9}{27}+\dfrac{16}{81}+\dfrac{25}{243}+$ &c., continued *ad infinitum*.

Let $x=\dfrac{1}{3}$ and $s=\dfrac{z}{(1-x)^3}$.

Then $\dfrac{z}{(1-x)^3}=x+4x^2+9x^3+16x^4+25x^5+$ &c.

And $z=(1-x)^3\times(x+4x^2+9x^3+16x^4$ &c.$)=x+x^2$, as will be found by actual multiplication.

Therefore $x+x^2=z$,

And $x+4x^2+9x^3+16x^4+$ &c. $=\dfrac{x(1+x)}{(1-x)^3}$.

Or

$\dfrac{1}{3}+\dfrac{4}{9}+\dfrac{9}{27}+\dfrac{16}{81}$ &c. $=\dfrac{\frac{1}{3}(1+\frac{1}{3})}{(1-\frac{1}{3})^3}=\dfrac{3}{2}=1\frac{1}{2}$, the sum required.

17. Required the sum (s) of the series $\dfrac{a}{m}+\dfrac{a+d}{mr}+\dfrac{a+2d}{mr^2}+\dfrac{a+3d}{mr^3}+$ &c., continued *ad infinitum*.

Let $x=\dfrac{1}{r}$, and $s=\dfrac{z}{m(1-x)^2}$.

Then $\dfrac{z}{m(1-x)^2}=\dfrac{a}{m}+\dfrac{a+d}{mr}+\dfrac{a+2d}{mr^2}+\dfrac{a+3d}{mr^3}+$ &c.

Or $\dfrac{z}{(1-x)^2}=a+\dfrac{a+d}{r}+\dfrac{a+2d}{r^2}+\dfrac{a+3d}{r^3}+$ &c.

That is, $\dfrac{z}{(1-x)^2}=$

$a+(a+d)x+(a+2d)x^2+(a+3d)x^3+(a+4d)x^4+$ &c.

And $z=(1-x)^2\times\{a+(a+d)x+(a+2d)x^2+(a+3d)x^3$
$+$ &c.$\}=(1-x)a+dx$,

as will appear by actual multiplication.

Therefore $z=(1-x)\,a+dx$; and consequently $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+$ &c. $=\frac{r}{m}\left\{\frac{a(r-1)+d}{(r-1)^2}\right\}$, the sum of the infinite series required.

EXAMPLES FOR PRACTICE.

1. Required the sum of 100 terms of the series 2, 5, 8, 11, 14, &c. *Ans.* 15050.

2. Required the sum of 50 terms of the series $1+2^2+3^2+4^2+5^2+$ &c. *Ans.* 42925.

3. It is required to find the sum of the series $1+3x+6x^2+10x^3+15x^4$ continued *ad infinitum*, &c., when x is a proper fraction, or less than 1. *Ans.* $\frac{1}{(1-x)^3}$.

4. It is required to find the sum of the series $1+4x+10x^2+20x^3+35x^4+$ &c., continued *ad infinitum*, when x is a fraction less than 1. *Ans.* $\frac{1}{(1-x)^4}$.

5. It is required to find the sum of the infinite series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+$ &c. *Ans.* $\frac{1}{2}$.

6. Required the sum of 40 terms of the series $(1\times2)+(3\times4)+(5\times6)+(7\times8)$ &c. *Ans.* 86920.

7. Required the sum of n terms of the series $\frac{2x-1}{2x}+\frac{2x-3}{2x}+\frac{2x-5}{2x}+\frac{2x-7}{2x}+$ &c. *Ans.* $n\left(\frac{2x-n}{2x}\right)$.

8. Required the sum of the infinite series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\frac{1}{4.5.6.7}+$ &c *Ans.* $\frac{1}{18}$.

9. Required the sum of the series $\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$ $+\frac{1}{35}+$ &c., continued *ad infinitum*. *Ans.* $\frac{3}{2}$, or $1\frac{1}{2}$.

10. It is required to find the sum of the infinite recurring series $1+8x+27x^2+64x^3+125x^4+$ &c.

Ans. $\frac{1+4x+x^2}{(1-x)^4}$.

11. Required the sum of n terms of the series $\frac{1}{r}+\frac{2}{r^2}$ $+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}+\frac{6}{r^6}+$ &c. *Ans.* $\frac{r}{(r-1)^2}-\frac{1}{r^n}\left\{\frac{nr+r-n}{(r-1)^2}\right\}$.

12. Required the sum of the series $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}$ $+\frac{1}{8.12}+$ &c. . . . . $+\frac{1}{2n(4+2n)}$.*

Ans. $\Sigma=\frac{3}{16}$, $s=\frac{5n+3n^2}{32+48n+16n^2}$.

13. Required the sum of the series $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}$ $+\frac{1}{12.20}+$ &c. . . . . $+\frac{1}{3n(4+4n)}$.

Ans. $\Sigma=\frac{1}{12}$, $s=\frac{n}{12+12n}$.

14. Required the sum of the series $\frac{6}{2.7}+\frac{6}{7.12}+\frac{6}{12.17}$ $+\frac{6}{17.22}+$ &c. . . . . $+\frac{6}{(5n-3).(5n+2)}$.

Ans. $\Sigma=\frac{3}{5}$, $s=\frac{3n}{5n+2}$.

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* The symbol Σ, made use of in these and some of the following series, denotes the sum of an infinite number of terms, and s the sum of n terms.

15. Required the sum of the series $\frac{1}{3.6}-\frac{1}{6.8}+\frac{1}{9.10}$ $-\frac{1}{12.12}+$ &c. . . . . $\pm\frac{1}{3n(4+2n)}$.

Ans. $\Sigma=\frac{1}{24}$, $s=\frac{n}{4(3+3n)}-\frac{n}{8(6+3n)}$, when n is *even*.

$s=\frac{n-1}{12n}-\frac{n-1}{24(n+1)}+\frac{1}{3n(4+2n)}$, when n is *odd*.

16. Required the sum of the series $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}$ $-\frac{5}{9.11}+$ &c. . . . . $\pm\frac{1+n}{(1+2n).(3+2n)}$.

Ans. $\Sigma=\frac{1}{12}$, $s=\frac{1}{12}\pm\frac{1}{4(3+2n)}$.

17. Required the sum of the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+$ $\frac{7}{3.4.5}+\frac{8}{4.5.6}+$ &c. . . . . $+\frac{4+n}{n(1+n)(2+n)}$.

Ans. $\Sigma=\frac{3}{2}$, $s=\frac{3}{2}-\frac{2}{1+n}+\frac{1}{2+n}$.*

OF LOGARITHMS.

(M) LOGARITHMS are a set of numbers that have been computed and formed into tables, for the purpose of facilitating arithmetical calculations; being so contrived, that the addition and subtraction of them answer to the mul-

* The series here treated of are such as are usually called algebraical; which, of course, embrace only a small part of the whole doctrine. Those, therefore, who may wish for further information on this abstruse, but highly curious subject, are referred to the *Miscellanea Analytica* of DEMOIVRE, STIRLING'S *Method Differ.*, JAMES BERNOULLI *de Seri. Infin.*, SIMPSON'S *Math. Dissert.*, WARING'S *Medit. Analyt.*, CLARKE'S translation of *Lorgna's Series*, the various works of EULER and LACROIX, *Traité du Calcul Diff. et Int.*, where they will find nearly all the materials that have been hitherto collected respecting this branch of analysis.

tiplication and division of the natural numbers, with which they are made to correspond.*

Or, when taken in a similar, but more general sense, logarithms may be considered as the indices, or exponents of the powers to which a given, or invariable, number must be raised, in order to produce all the common, or natural numbers. Thus, if

$$a^x=y,\ a^{x'}=y',\ a^{x''}=y'', \text{ \&c.,}$$

then will the indices x, x', x'', &c., of the several powers of a, be the logarithms of the numbers y, y', y'', &c., in the scale, or system, of which a is the base.

So that, from either of these formulæ, it appears that the logarithm of any number, taken separately, is the index of that power of some other number, which, when involved in the usual way, is equal to the given number.

And since the base a, in the above expressions, can be assumed of any value, greater or less than 1, it is plain that there may be an endless variety of systems of logarithms, answering to the same natural numbers.

It is, likewise, further evident, from the first of these equations, that when $y=1$, x will be $=0$, whatever may

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* This mode of computation, which is one of the happiest and most useful discoveries of modern times, is due to LORD NAPIER, Baron of Merchiston, in Scotland, who first published a table of these numbers, in the year 1614, under the title of *Canon Mirificum Logarithmorum;* which performance was eagerly received by the learned throughout Europe, whose efforts were immediately directed to the improvement and extension of the new calculus, that had so unexpectedly presented itself.

MR. HENRY BRIGGS, in particular, who was, at that time, Professor of Geometry in Gresham College, greatly contributed to the advancement of this doctrine, not only by the very advantageous alteration which he first introduced into the system of these numbers, by making 1 the logarithm of 10, instead of 2.3025852..., as has been done by NAPIER; but also by the publication, in 1624 and 1633, of his two great works, the *Arithmetica Logarithmica*, and the *Trigonometria Britannica*, both of which were formed upon the principle above mentioned; as are, likewise, all our common logarithmic tables at present in use.

See, for farther details on this part of the subject, the introduction to my *Treatise of Plane and Spherical Trigonometry*, 8vo. 3d Edit. 1818; and for the construction and use of the tables, consult those of SHERWIN, HUTTON, TAYLOR, CALLET, and BORDA, where every necessary information of this kind may be readily obtained.

be the value of a; and consequently the logarithm of 1 is always 0, in every system of logarithms.

And if $x=1$, it is manifest, from the same equation, that the base a will be $=y$; which base is, therefore, the number whose proper logarithm, in the system to which it belongs, is 1.

Also, because $a^x=y$, and $a^{x'}=y'$, it follows, from the multiplication of powers, that $a^x \times a^{x'}$, or $a^{x+x'}=yy'$; and consequently, by the definition of logarithms, given above, $x+x'=\log. yy'$, or

$$\log. yy' = \log. y + \log. y'.$$

And, for a like reason, if any number of the equations $a^x=y, a^{x'}=y', a^{x''}=y''$, &c., be multiplied together, we shall have $a^{x+x'+x'', \&c.} = yy' y''$, &c.; and, consequently, $x+x'+x''$, &c. $= \log. yy' y''$, &c.; or

$$\log. yy' y'', \&c., = \log. y + \log. y' + \log. y'', \&c.$$

From which it is evident that the logarithm of the product of any number of factors is equal to the sum of the logarithms of those factors.

Hence, if all the factors of a given number, in any case of this kind, be supposed equal to each other, and the sum of them be denoted by m, the preceding property will then become

$$\log. y^m = m \log. y.$$

From which it appears that the logarithm of the mth power of any number is equal to m times the logarithm of that number.

In like manner, if the equation $a^x=y$ be divided by $a^{x'}=y'$, we shall have, from the nature of powers, as before, $\frac{a^x}{a^{x'}}$, or $a^{x-x'} = \frac{y}{y'}$; and by the definition of logarithms, laid down in the first part of this article, $x - x' = \log. \frac{y}{y'}$, or

$$\log. \frac{y}{y'} = \log. y - \log. y'.$$

Hence the logarithm of a fraction, or of the quotient arising from dividing one number by another, is equal to the logarithm of the numerator *minus* the logarithm of the denominator, or to the logarithm of the dividend *minus* that of the divisor.

And if each member of the common equation $a^x=y$ be

raised to the fractional power denoted by $\frac{m}{n}$, we shall have in that case, $a^{\frac{m}{n}x} = y^{\frac{m}{n}}$;

And, consequently, by taking the logarithms of these quantities, as before, $\frac{m}{n}x = \log. y^{\frac{m}{n}}$, or

$$\log. y^{\frac{m}{n}} = \frac{m}{n}\log. y.$$

Where it appears, that the logarithm of a mixed root, or power, of any number, is found by multiplying the logarithm of the given number by the numerator of the index of that power, and dividing the result by the denominator.

And if the numerator m of the fractional index be, in this case, taken equal to 1, the above formula will then become

$$\log. y^{\frac{1}{n}} = \frac{1}{n}\log. y.$$

From which it follows, that the logarithm of the nth root of any number is equal to the nth part of the logarithm of that number.

Hence, besides the use of logarithms in abridging the operations of multiplication and division, they are equally applicable to the raising of powers, and extracting of roots; which are performed by simply multiplying the given logarithm by the index of the power, or dividing it by the number denoting the root.

But, although the properties here mentioned are common to every system of logarithms, it was necessary, for practical purposes, to select some one of them from the rest, and to adapt the logarithms of all the natural numbers to that particular scale.

And as 10 is the base of our present system of arithmetic, the same number has accordingly been chosen for the base of the logarithmic system, now generally used.

So that, according to this scale, which is that of the common logarithmic tables, the numbers

$$\ldots\ 10^{-4},\ 10^{-3},\ 10^{-2},\ 10^{-1},\ 10^{0},\ 10^{1},\ 10^{2},\ 10^{3},\ 10^{4},\ \&c.$$

$$\text{Or}\ \ldots\ \frac{1}{10000},\ \frac{1}{1000},\ \frac{1}{100},\ \frac{1}{10},\ 1,\ 10,\ 100,\ 1000,\ 10000, \&c.,$$

have for their logarithms

. . $-4, -3, -2, -1, 0, 1, 2, 3, 4$, &c.

Which are evidently a set of numbers in arithmetical progression, answering to another set in geometrical progression; as is the case in every system of logarithms.*

And therefore, since the common, or tabular, logarithm of any number (n) is the index of that power of 10, which, when involved, is equal to the given number, it is plain, from the following equation,

$$10^x = n, \text{ or } 10^{-x} = \frac{1}{n},$$

that the logarithms of all the intermediate numbers, in the above series, may be assigned by approximation, and made to occupy their proper places in the general scale.

It is also evident, that the logarithms of 1, 10, 100, 1000, &c., being 0, 1, 2, 3, &c., respectively, the logarithm of any number, falling between 1 and 10, will be 0 and some decimal parts; that of a number between 10 and 100, 1 and some decimal parts; of a number between 100 and 1000, 2 and some decimal parts; and so on, for other numbers of this kind.

In like manner, the logarithms of $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$, &c., or of their equals, ·1, ·01, ·001, &c., in the descending part of the scale, being $-1, -2, -3$, &c., the logarithm of any number, falling between 0 and ·1, will be -1 and some positive decimal parts; that of a number between 1 and ·01, -2 and some positive decimal parts; of a number between ·01 and ·001, -3 and some positive decimal parts; &c.

Hence also, as the multiplying or dividing of any number by 10, 100, 1000, &c., is performed by barely increasing or diminishing the integral part of its logarithm by 1, 2, 3, &c., it is obvious that all numbers, which consist of the same figures, whether they be integral, fractional, or mixed, will have, for the decimal part of their logarithms, the same positive quantity.

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* A detailed account of the theory and construction of logarithms may be seen in Vol. II. of my *Treatise on Algebra*. Also is just published a very useful and well-written Treatise on the Computation of Logarithms, showing the most expeditious methods for constructing Tables of Logarithms, &c. By Professor Young, *of Belfast College.*

So that, in this system, the integral part of any logarithm, which is usually called its index, or characteristic, is always less by 1 than the number of integers which the natural number consists of; and for decimals, it is the number which denotes the distance of the first significant figure from the place of units.

Thus, according to the logarithmic tables in common use, we have

Numbers.	*Logarithms.*
1·36820	0·1361496
20·0500	1·3021144
335·260	2·5253817
·46521	$\bar{1}$·6676490
·06154	$\bar{2}$·7891575
&c.	&c.;

Where the sign − is put over the index, instead of before it, when that part of the logarithm is negative, in order to distinguish it from the decimal part, which is always to be considered as +, or affirmative.

Also, agreeably to what has been before observed, the logarithm of 38540 being 4·5859117, the logarithms of any other numbers, consisting of the same figures, will be as follow:

Numbers.	*Logarithms.*
3854	3·5859117
385·4	2·5859117
38·54	1·5859117
3·854	0·5859117
·3854	$\bar{1}$·5859117
·03854	$\bar{2}$·5859117
·003854	$\bar{3}$·5859117

Which logarithms, in this case, as well as in all others of a similar kind, whether the number contains ciphers or not, differ only in their indices, the decimal or positive part being the same in them all.*

* The great advantages attending the common, or BRIGGEAN system of logarithms above all others, arise chiefly from the readiness

And as the indices, or integral parts of the logarithms of any numbers whatever, in this system, can always be thus readily found, from the simple consideration of the rule above mentioned, they are generally omitted in the tables, being left to be supplied by the operator, as occasion requires.

It may here, also, be further added, that, when the logarithm of a given number, in any particular system, is known, it will be easy to find the logarithm of the same number in any other system, by means of the following equations:

$$a^x = n, \text{ and } e^{x'} = n; \text{ or log. } n = x, \text{ and } l.n = x'.$$

Where log. denotes the logarithm of n, in the system of which a is the base, and $l.$ its logarithm in the system of which e is the base.

For, since $a^x = e^{x'}$, or $a^{\frac{x}{x'}} = e$, and $e^{\frac{x'}{x}} = a$, we shall have, for the base a, $\frac{x}{x'} =$ log. e, or $x = x'$ log. e;

and for the base e, $\frac{x'}{x} = l.a$, or $x' = xl.a$.

Whence, by substitution, from the former equations

$$\text{log. } n = l.n \times \text{log. } e; \text{ or log. } n = l.n \times \frac{1}{l.a}.$$

Where the multiplier log. e, or its equal $\frac{1}{l.a}$, expresses the constant relation which the logarithms of n have to each other in the systems to which they belong.

But the only system of these numbers, deserving of notice, except that above described, is the one that furnishes what

with which we can always find the characteristic or integral part of any logarithm from the bare inspection of the natural number to which it belongs; and the circumstance, that multiplying or dividing any number by 10, 100, 1000, &c., only influences the characteristic of its logarithm, without affecting the decimal part. Thus, for instance, if i be made to denote the index, or integral part of the logarithm of any number N, and d its decimal part, we shall have

$$\text{Log. N} = i + d; \text{ log. } 10^m\text{N} = (i+m)+d; \text{ log. } \frac{\text{N}}{10^m} = (i-m)+d;$$

where it is plain that the decimal part of the logarithm, in each of these cases, remains the same.

have been usually called hyperbolic or NAPIERIAN logarithms, the base e of which is 2·718281828459 . . . .

Hence, in comparing these with the common or tabular logarithms, we shall have, by putting a in the latter of the above formulæ $= 10$, the expression

$$\log.\ n = l.n \times \frac{1}{l.10}, \text{ or } l.n = \log.\ n \times l.10,$$

Where log. in this case denotes the common tabular logarithm of the number n, and l its hyperbolic logarithm; the constant factor, or multiplier, $\frac{1}{l.10}$, which is

$$\frac{1}{2\cdot 302585092\overline{9}}, \text{ or its equal } \cdot 4342944819,$$

being what is usually called the *modulus* of the common system of logarithms.*

PROBLEM I.

To compute the logarithm of any of the natural numbers, 1, 2, 3, 4, 5, &c.

RULE I.

1. Take the geometrical series, 1, 10, 100, 1000, 10000, &c., and apply to it the arithmetical series, 0, 1, 2, 3, 4, &c., as logarithms.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the series, betwixt which the number proposed lies.

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* It may here be remarked, that although the common logarithms have superseded the use of hyperbolic or NAPIERIAN logarithms, in all the ordinary operations to which these numbers are generally applied, yet the latter are not without some advantages peculiar to themselves; being of frequent occurrence in the application of the Fluxionary Calculus to many analytical and physical problems, where they are required for the finding of certain fluents, which could not be so readily determined without their assistance; on which account, great pains have been taken to calculate tables of hyperbolic logarithms, to a considerable extent, chiefly for this purpose. Mr. BARLOW, in a *Collection of Mathematical Tables* lately published, has given them for the first 10,000 numbers.

3. Also, between the mean, thus found, and the nearest extreme, find another geometrical mean, in the same manner; and so on, till you are arrived within the proposed limit of the number whose logarithm is sought.

4. Find, likewise, as many arithmetical means between the corresponding terms of the other series, 0, 1, 2, 3, 4, &c., in the same order as the geometrical means were found, and the last of these will be the logarithm answering to the number required.

EXAMPLES.

1. Let it be required to find the logarithm of 9.

Here the proposed number lies between 1 and 10.

First, then, the log. of 10 is 1, and the log. of 1 is 0.

Also $\sqrt{(10 \times 1)} = \sqrt{10} = 3\cdot1622777$ is the geometrical mean;

And $\frac{1}{2}(1+0) = \frac{1}{2} = \cdot5$ is the arithmetical mean;

Hence the log. of $3\cdot1622777$ is $\cdot5$.

Secondly, the log. of 10 is 1, and the log of $3\cdot1622777$ is $\cdot5$.

Therefore $\sqrt{(10 \times 3\cdot1622777)} = 5\cdot6234132$ is the geometrical mean;

And $\frac{1}{2}(1 + \cdot5) = \cdot75$ is the arithmetical mean;

Hence the log. of $5\cdot6234132$ is $\cdot75$.

Thirdly, the log. of 10 is 1, and the log of $5\cdot6234132$ is $\cdot75$;

Therefore $\sqrt{(10 \times 5\cdot6234132)} = 7\cdot4989422$ is the geometrical mean;

And $\frac{1}{2}(1 \times \cdot75) = \cdot875$ is the arithmetical mean;

Hence the log. of $7\cdot4989422$ is $\cdot875$.

Fourthly, the log. of 10 is 1, and the log. of $7\cdot4989422$ is $\cdot875$;

Therefore $\sqrt{(10 \times 7\cdot4989422)} = 8\cdot6596431$ is the geometrical mean,

And $\frac{1}{2}(1 + \cdot875) = \cdot9375$ is the arithmetical mean;

Hence the log. of $8\cdot6596431$ is $\cdot9375$.

Fifthly, the log. of 10 is 1, and the log. of $8\cdot6596431$ is $\cdot9375$.

Therefore $\sqrt{(10 \times 8{\cdot}6596431)} = 9{\cdot}3057204$ is the geometrical mean.

And $\frac{1}{2}(1 + .9375) = {\cdot}96875$ is the arithmetical mean;

Hence the log. of 9·3057204 is ·96875.

Sixthly, the log. of 8·6596431 is ·9375, and the log. of 9·3057204 is ·96875;

Therefore $\sqrt{(8{\cdot}6596431 \times 9{\cdot}3057204)} = 8{\cdot}9768713$ is the geometrical mean.

And $\frac{1}{2}({\cdot}9375 + {\cdot}96875) = {\cdot}953125$ is the arithmetical mean;

Hence the log. of 8·9768713 is ·953125.

And, by proceeding in this manner, it will be found, after 25 extractions, that the logarithm of 8·9999998 is ·9542425; which may be taken for the logarithm of 9, as it differs from it so little, that it may be considered as sufficiently exact for all practical purposes.

And in this manner were the logarithms of all the prime numbers at first computed.

RULE II.

When the logarithm of any number (n) is known, the logarithm of the next greater number may be found from the following series, by calculating a sufficient number of its terms, and then adding the given logarithm to their sum.

$$\text{Log.}\ (n+1) = \log.\ n + 2\text{M}\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \frac{1}{7(2n+1)^7} + \frac{1}{9(2n+1)^9} + \frac{1}{11(2n+1)^{11}} + \&\text{c.}\right\}^*$$

* Dem. $\text{Log.}\ \overline{n+1} = \log.\ n.\ (1 + \frac{1}{n})$

$= \log.\ n + \log.\ (1 + \frac{1}{n})$

let $1 + \frac{1}{n} = \frac{1+y}{1-y}\ \therefore y = \frac{1}{2n+1}$.

Now

or

$$\text{Log.}\,(n+1) = \log.\, n + \left\{ \frac{2\text{M}}{2n+1} + \frac{\text{A}}{3(2n+1)^2} + \frac{3\text{B}}{5(2n+1)^2} \right.$$

$$\left. + \frac{5\text{C}}{7(2n+1)^2} + \frac{7\text{D}}{9(2n+1)^2} + \frac{9\text{E}}{11(2n+1)^2} + \&c. \right\}$$

Where A, B, C, &c., represent the terms immediately preceding those in which they are first used; the modulus M = ·4342944819 . . . ; and consequently, 2M = ·8685889638 . . . *

EXAMPLES.

1. Let it be required to find the common logarithm of the number 2.

Here, because $n + 1 = 2$, and consequently $n = 1$ and $2n + 1 = 3$, we shall have

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$$\text{Now } \log.\ 1+y = M\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \&c.\right)$$

$$\log.\ 1-y = M\left(-y - \frac{y^2}{2} - \frac{y^3}{3} - \&c.\right)$$

$$\therefore \log.\ \frac{1+y}{1-y} = 2M\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \&c.\right)$$

$$\text{Or } \log.\left(1 + \frac{1}{n}\right) = 2M\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \&c.\right)$$

$$\therefore \log.\,(n+1) = \log.\, n + 2M\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5}\right.$$

$$\left. + \frac{1}{7(2n+1)^7} + \&c.\right)$$

* It may be here remarked, that the difference between the logarithms of any two consecutive numbers is so much the less as the numbers are greater; and consequently the series which comprises the latter part of the above expression, will in that case converge so much the faster. Thus

$$\log.\ n, \text{ and } \log.\ (n+1), \text{ or its equal } \log.\ n + \log.\ \left(1 + \frac{1}{n}\right)$$

will, obviously, differ but little from each other when n is a large number.

$$\frac{2\text{M}}{2n+1}=\frac{\cdot8685889638}{3}=\cdot289529655\ (\text{A})$$

$$\frac{\text{A}}{3(2n+1)^2}=\frac{\cdot289529655}{3.3^2}=\cdot010723321\ (\text{B})$$

$$\frac{3\text{B}}{5(2n+1)^2}=\frac{3\times\cdot010723321}{5.3^2}=\cdot000714888\ (\text{C})$$

$$\frac{5\text{C}}{7(2n+1)^2}=\frac{5\times\cdot000714888}{7.3^2}=\cdot000056737\ (\text{D})$$

$$\frac{7\text{D}}{9(2n+1)^2}=\frac{7\times\cdot000056737}{9.3^2}=\cdot000004903\ (\text{E})$$

$$\frac{9\text{E}}{11(2n+1)^2}=\frac{9\times\cdot000004903}{11.3^2}=\cdot000000446\ (\text{F})$$

$$\frac{11\text{F}}{13(2n+1)^2}=\frac{11\times\cdot000000446}{13.3^2}=\cdot000000042\ (\text{G})$$

$$\frac{13\text{G}}{15(2n+1)^2}=\frac{13\times\cdot000000042}{15.3^2}=\cdot000000004\ (\text{H})$$

Sum of 8 terms . .	·301029996
Add log. of 1 . . .	·000000000
Log. of 2	·301029996

Which logarithm is true to the last figure inclusively.

2. Let it be required to compute the logarithm of the number 3.

Here, since $n+1=3$, and consequently $n=2$, and $2n+1=5$, we shall have

$$\frac{2\text{M}}{2n+1} = \frac{\cdot 868588964}{5} \quad . \; . = \cdot 173717793 \; (\text{A})$$

$$\frac{\text{A}}{3(2n+1)^2} = \frac{\cdot 173717793}{3.5^2} \quad . \; . = \cdot 002316237 \; (\text{B})$$

$$\frac{3\text{B}}{5(2n+1)^2} = \frac{3 \times \cdot 002316237}{5.5^2} \quad . \; . = \cdot 000055590 \; (\text{C})$$

$$\frac{5\text{C}}{7(2n+1)^2} = \frac{5 \times \cdot 000055590}{7.5^2} \quad . \; . = \cdot 000001588 \; (\text{D})$$

$$\frac{7\text{D}}{9(2n+1)^2} = \frac{7 \times \cdot 000001588}{9.5^2} \quad . \; . = \cdot 000000049 \; (\text{E})$$

$$\frac{9\text{E}}{11(2n+1)^2} = \frac{9 \times \cdot 000000049}{11.5^2} \quad . \; . = \cdot 000000002 \; (\text{F})$$

Sum of 6 terms	·176091259
Log. of 2	·301029996
Log. of 3	·477121255

Which logarithm is also correct to the nearest unit in the last figure.

And in the same way we may proceed to find the logarithm of any prime number.

Also, because the sum of the logarithms of any two numbers gives the logarithm of their product, and the difference of the logarithms the logarithm of their quotient, &c.; we may readily compute, from the above two logarithms, and the logarithm of 10, which is 1, a great number of other logarithms, as in the following examples:

3. Because $2^2 = 4$, therefore log. 2	·301029996
mult. by 2	2
gives log. 4	·602059992

4. Because $2 \times 3 = 6$, therefore to log. 2	·301029996
add log. 3	·477121255
gives log. 6	·778151251

5. Because $2^3 = 8$, therefore	log. 2	·301029996
	mult. by 3	3
	gives log. 8	·903089988

6. Because $3^2 = 9$, therefore	log. 3	·477121255
	mult. by 2	2
	gives log. 9	·954242510

7. Because $\frac{10}{2} = 5$, therefore	from log. 10	1·000000000
	take log. 2	·301029996
	gives log. 5	·698970004

8. Because $3 \times 4 = 12$, therefore	to log. 3	·477121255
	add log. 4	·602059992
	gives log. 12	1·079181247

And thus, by computing, according to the general formula, the logarithms of the next succeeding prime numbers, 7, 11, 13, 17, 19, 23, &c., we can find, by means of the simple rules before laid down for multiplication, division, and the raising of powers, as many other logarithms as we please, or may speedily examine any logarithm in the table.

MULTIPLICATION BY LOGARITHMS.

Take out the logarithms of the factors from the table, and add them together; then the natural number answering to the sum will be the product required.

Observing, in the addition, that what is to be carried from the decimal part of the logarithms is always affirmative, and must, therefore, be added to the indices, or integral parts, after the manner of positive and negative quantities in algebra.

This method will be found much more convenient, to those who possess a slight knowledge of this science, than that of using the arithmetical complements.

EXAMPLES.

1. Multiply 37·153 by 4·086, by logarithms.

Nos.	*Logs.*
37·153	1·5699939
4·086	0·6112984
Prod. 151·8072 . .	2·1812923

2. Multiply 112·246 by 13·958, by logarithms.

Nos.	*Logs.*
112·246	2·0501709
13·958	1·1448232
Prod. 1566·730 . . .	3·1949941

3. Multiply 46·7512 by ·3275, by logarithms.

Nos.	*Logs.*
46·7512	1·6697928
·3275	$\bar{1}$·5152113
Prod. 15·31102 . . .	1·1850041

Here, the + 1, that is to be carried from the decimals, cancels the − 1, and consequently there remains 1 in the upper line to be set down.

4. Multiply ·37816 by ·04782, by logarithms.

Nos.	*Logs.*
·37816	$\bar{1}$·5776756
·04782	$\bar{2}$·6796096
Prod. 0·0180836 . .	$\bar{2}$·2572852

Here the + 1 that is to be carried from the decimals, destroys the − 1, in the upper line, as before, and there remains the − 2 to be set down.

5. Multiply 3·768, 2·053, and ·007693 together.

Nos.	*Logs.*
3·768	0·5761109
2·053	0·3123889
·007693	$\bar{3}$·8860957
Prod. ·059511 . . .	$\bar{2}$·7745955

Here the + 1 that is to be carried from the decimals, when added to − 3, makes − 2, to be set down.

6. Multiply 3·586, 2·1046, ·8372, and ·0294, together.

Logs.	*Nos.*
3·586	0·554610
2·1046	0·323170
·8372	$\bar{1}$·922829
·0294	$\bar{2}$·468347
Prod. ·1857618 . . .	$\bar{1}$·268956

Here the + 2, that is to be carried, cancels the −2, and there remains the −1 to be set down.

7. Multiply 23·14 by 5·062, by logarithms.
Ans. 117·1347.

8. Multiply 4·0763 by 9·8432, by logarithms.
Ans. 40·12383

9. Multiply 498·256 by 41 2467, by logarithms.
Ans. 20551·41.

10. Multiply 4·26747 by ·012345, by logarithms.
Ans. ·05268191.

11. Multiply 3·12567, ·02868 and ·12379 together, by logarithms. *Ans.* ·01109706.

12. Multiply 2876·9, ·10674, ·098762 and ·0031598, by logarithms. *Ans.* ·09583.

DIVISION BY LOGARITHMS.

From the logarithm of the dividend, as found in the tables, subtract the logarithm of the divisor, and the natural number, answering to the remainder, will be the quotient required.

Observing, if the subtraction cannot be made in the usual way, to add, as in the former rule, the 1 that is to be carried from the decimal part, when it occurs, to the index of the logarithm of the divisor, and then this result with its sign changed, to the remaining index, for the index of the logarithm of the quotient.

EXAMPLES.

1. Divide 4768·2 by 36·954, by logarithms.

Nos.	*Logs.*
4768·2	3·6783545
36·954	1·5676615
Quot. 129·0307 . . .	2·1106930

2. Divide 21·754 by 2·4678, by logarithms.

Nos.	*Logs.*
21·754	1·3375391
2·4678	0·3923100
Quot. 8·81514 . . .	0·9452291

3. Divide 4·6257 by ·17608, by logarithms.

Nos.	*Logs.*
4·6257	0·6651775
·17608	$\bar{1}$·2457100
Quot. 26·2704 . . .	1·4194675

Here − 1, in the lower index, is changed into + 1, which is then taken for the index of the result.

4. Divide ·27684 by 5·1576, by logarithms.

Nos.	*Logs.*
·27684	$\bar{1}$·4422288
5·1576 . . , .	0·7124477
Quot. ·0536761 . .	$\bar{2}$·7297811

Here the 1 that is to be carried from the decimals, is taken as -1, and then added to -1 in the upper index which gives -2 for the index of the result.

5. Divide 6·9875 by ·075789, by logarithms.

Nos.	*Logs.*
6·9875	0·8443218
·075789	$\bar{2}$·8796062
Quot. 92·1967 . . .	1·9647156

Here the 1, that is to be carried from the decimals, is added to -2, which makes -1, and this put down, with its sign changed, is $+1$.

6. Divide ·19876 by ·0012345, by logarithms.

Nos.	*Logs.*
·19876	$\bar{1}$·2983290
·0012345 . . .	$\bar{3}$·0914911
Quot. 161·0044 . .	2·2068379

Here -3, in the lower index, is changed into $+3$ and this added to -1, the other index, gives $+3-1$ or 2.

7. Divide 125 by 1728, by logarithms

Ans. ·0723379.

8. Divide 1728·95 by 1·10678, by logarithms.

Ans. 1562·145.

9. Divide 10·23674 by 4·96523, by logarithms.

Ans. 2·061686.

10. Divide 19965·7 by ·048235, by logarithms.
Ans. 413926.

11. Divide ·067859 by 1234·59, by logarithms.
Ans. ·0000549648.

THE RULE OF THREE,

OR PROPORTION, BY LOGARITHMS.

For any single proportion, add the logarithms of the second and third terms together, and subtract the logarithm of the first from their sum, according to the foregoing rules; then the natural number answering to the result will be the fourth term required.

But if the proportion be compound, add together the logarithms of all the terms that are to be multiplied, and from the result take the sum of the logarithms of the other terms, and the remainder will be the logarithm of the terms sought.

Or the same may be performed more conveniently, thus,—

Find the complement of the logarithm of the first term of the proportion, or what it wants of 10, by beginning at the left hand, and taking each of its figures from 9, except the last significant figure on the right, which must be taken from 10; then add this result and the logarithms of the other two terms together, and the sum, abating 10 in the index, will be the logarithm of the fourth term, as before.

And, if two or more logarithms are to be subtracted, as in the latter part of the above rule, add their complements and the logarithms of the terms to be multiplied together, and the result, abating as many 10s in the index as there are logarithms to be subtracted, will be the logarithm of the term required; observing, when the index of the logarithm, whose complement is to be taken, is negative, to add it, as if it were affirmative, to 9; and then take the rest of the figures from 9, as before.

EXAMPLES.

1. Find a fourth proportional to 37·125, 14·768 and 135·279, by logarithms.

Log. of 37·125	1·5696665
Complement	8·4303335
Log. of 14·768	1·1693217
Log. of 135·279	2·1312304
Ans. 53·8128	1·7308856

2. Find a fourth proportional to ·05764, ·7186 and ·34721, by logarithms.

Log. of ·05764 . . .	$\bar{2}$·7607240
Complement . . .	11·2392760
Log. of ·7186 . . .	$\bar{1}$·8564872
Log. of ·34721 . . .	$\bar{1}$·5405922
Ans. 4·328678 . . .	0·6363554

3. Find a third proportional to 12·796 and 3·24718, by logarithms.

Log. of 12·796 . . .	1·1070742
Complement	8·8929258
Log. of 3·24718 . . .	0·5115063
Log. of 3·24718 . . .	0·5115063
Ans. ·8240211 . .	$\bar{1}$·9159384

4. Find the interest of 279*l.* 5*s.* for 274 days, at 4½ per cent. per annum, by logarithms.

Comp. log. of 100 . .	8·0000000
Comp. log. of 365 . .	7·4377071
Log. of 279·25 . . .	2·4459932
Log. of 274	2·4377506
Log. of 4·5	0·6532125
Ans. 9·433297 . .	0·9746634

5. Find a fourth proportional to 12·678, 14·065 and 100·979, by logarithms. *Ans.* 112·0263.

6. Find a fourth proportional to 1·9864, ·4678 and 50·4567, by logarithms. *Ans.* 11·88262.

7. Find a fourth proportional to ·09658, ·24958 and ·008967, by logarithms. *Ans.* ·02317234.

8. Find a third proportional to ·498621 and 2·9587, and a third proportional to 12·796 and 3·24718, by logarithms. *Ans.* 17·55623 and ·8240216.

INVOLUTION,

OR THE RAISING OF POWERS, BY LOGARITHMS.

Take out the logarithm of the given number from the tables, and multiply it by the index of the proposed power; then the natural number, answering to the result, will be the power required.

Observing, if the index of the logarithm be negative, that this part of the product will be negative; but as what is to be carried from the decimal part will be affirmative, the index of the result must be taken accordingly.

EXAMPLES.

1. Find the square of 2·7558, by logarithms.

Log. of 2·7558 . . .	0·4402477
	2
Square 7·594434 . .	0·8804954

2. Find the cube of 7·0851, by logarithms.

Log. of 7·0851 . . .	0·8503460
	3
Cube 355·6625 . .	2·5510380

3. Find the fifth power of ·87451, by logarithms.

Log. ·87451 . .	$\overline{1}$·9417648
	5
Fifth power ·5114745 .	$\overline{1}$·7088240

Where 5 times the negative index $\overline{1}$, being -5, and $+4$ to carry, the index of the power is $\overline{1}$.

4. Find the 365th power of 1·0045, by logarithms.

Log. 1·0045 . . .	0·0019499
	365
	97495
	116994
	58497
365th power 5·148888 Log.	0·7117135

5. Required the square of 6·05987, by logarithms.
Ans. 36·72203.

6. Required the cube of ·176546, by logarithms.
Ans. ·005502674.

7. Required the 4th power of ·076543, by logarithm.
Ans. ·00003432591.

8. Required the 5th power of 2·97643, by logarithms.
Ans. 233·6031.

9. Required the 6th power of 21·0576, by logarithms.
Ans. 87187340.

10. Required the 7th power of 1·09684, by logarithms.
Ans. 1·909864.

EVOLUTION,

OR THE EXTRACTION OF ROOTS, BY LOGARITHMS.

Take out the logarithm of the given number from the table, and divide it by 2 for the square root, 3 for the cube

root, &c., and the natural number answering to the result will be the root required.

But if it be a compound root, or one that consists both of a root and a power, multiply the logarithm of the given number by the numerator of the index, and divide the product by the denominator, for the logarithm of the root sought.

Observing, in either case, when the index of the logarithm is negative, and cannot be divided without a remainder, to increase it by such a number as will render it exactly divisible; and then carry the units borrowed, as so many tens, to the first figure of the decimal part, and divide the whole accordingly.

EXAMPLES.

1. Find the square root of 27·465, by logarithms.

Log. of 27·465 . .	2)1·4387796
Root 5·24076	·7193898

2. Find the cube root of 35·6415, by logarithms

Log. of 35·6415 . .	3)1·5519560
Root 3·29093	·5173187

3. Find the 5th root of 7·0825, by logarithms.

Log. of 7·0825 . .	5)0·8501866
Root 1·479235 . . .	·1700373

4. Find the 365th root of 1·045, by logarithms.

Log. of 1·045 . . .	365)0.0191163
Root 1·000121	0·0000524

5. Find the value of $(\cdot 001234)^{\frac{3}{2}}$ by logarithms.

$$\begin{array}{lr} \text{Log. of } \cdot 001234 \quad . \quad . & \bar{3}\cdot 0913152 \\ & 2 \\ \hline & 3)\bar{6}\cdot 1826304 \\ \hline \textit{Ans. } \cdot 0115047 \quad . \quad . \quad . \quad . & \bar{2}\cdot 0608768 \\ \hline \end{array}$$

Here, the divisor 3 being contained exactly twice in the negative index -6, the index of the quotient, to be put down, will be -2.

6. Find the value of $(\cdot 024554)^{\frac{3}{2}}$, by logarithms.

$$\begin{array}{lr} \text{Log. of } \cdot 024554 \quad . \quad . \quad . \quad . & \bar{2}\cdot 3901223 \\ & 3 \\ \hline & 2)\bar{5}\cdot 1703669 \\ \hline \textit{Ans. } \cdot 00384754 \quad . \quad . \quad . \quad . & \bar{3}\cdot 5851834 \\ \hline \end{array}$$

Here 2 not being contained exactly in -5, 1 is added to it, which gives -3 for the quotient; and the 1 that is borrowed being carried to the next figure, makes 11, which, divided by 2, gives ·5851834 for the decimal part of the logarithm.

7. Required the square root of 365·5674, by logarithms.
Ans. 19·11981.

8. Required the cube root of 2·987635, by logarithms.
Ans. 1·440265.

9. Required the 4th root of ·967845, by logarithms.
Ans. ·9918624.

10. Required the 7th root of ·098674, by logarithms.
Ans. ·7183146.

11. Required the value of $\left(\frac{21}{373}\right)^{\frac{2}{3}}$, by logarithms.
Ans. ·146895

12. Required the value of $\left(\frac{112}{1727}\right)^{\frac{3}{5}}$, by logarithms.
Ans. ·1937115.

MISCELLANEOUS EXAMPLES IN LOGARITHMS.

1. Required the square root of $\frac{2}{123}$, by logarithms.

Ans. ·1275153.

2. Required the cube root of $\frac{1}{3·14159}$, by logarithms.

Ans. ·6827842.

3. Required the ·07 power of ·00563, by logarithms.

Ans. ·6958818.

4. Required the value of $\frac{(\frac{2}{3})^{\frac{1}{2}} \times (\frac{3}{4})^{\frac{1}{3}}}{17\frac{1}{3}}$, by logarithms.

Ans. ·04279826.

5. Required the value of $\frac{1}{7}\sqrt{\frac{5}{8}} \times {·012}\sqrt[3]{\frac{7}{11}}$, by logarithms.

Ans. ·001165713.

6. Required the value of $\frac{\frac{1}{9}\sqrt{\frac{11}{21}} \times ·03\sqrt[3]{15\frac{1}{5}}}{7\frac{1}{3}\sqrt[3]{12\frac{1}{5}} \times ·19\sqrt[4]{17\frac{1}{8}}}$, by logarithms.

Ans. ·0009158638.

7. Required the value of $\frac{127}{4}\left(\frac{\frac{5}{6}\sqrt{19}+\frac{4}{7}\sqrt[3]{35\frac{1}{3}}}{14\frac{7}{19}-\frac{15}{11}\sqrt{28\frac{2}{3}}}\right)$, by logarithms.

Ans. 24·7447.

MISCELLANEOUS QUESTIONS.

1. A person being asked what o'clock it was, said it is between eight and nine, and the hour and minute hands are exactly together; what was the time?

Ans. 8h. 43 min. $38\frac{2}{11}$ sec.

2. A certain number, consisting of two places of figures, is equal to the difference of the squares of its digits, and if 36 be added to it the digits will be inverted; what is the number?

Ans. 48.

3. What two numbers are those, whose difference, sum, and product, are to each other as the numbers 2, 3, and 5, respectively?

Ans. 2 and 10.

4. A person in a party at cards, betted three shillings to two upon every deal, and after twenty deals found he had gained five shillings; how many deals did he win?

Ans. 13.

5. A person wishing to inclose a piece of ground with palisades, found, if he set them a foot asunder, that he should have too few by 150, but if he set them a yard asunder, he should have too many by 70; how many had he? *Ans.* 180.

6. A cistern will be filled by two cocks, A and B, running together, in twelve hours, and by the cock A alone in twenty hours, in what time will it be filled by the cock B alone?

Ans. 30 hrs.

7. A grocer bought a lot of tea at 10*s.* a lb., and a quantity of coffee at 2*s.* 6*d.* a lb., which cost him altogether 31*l.* 5*s.*: but the state of the market having changed, he sold the tea at 8*s.* a lb. and the coffee at 4*s.* 6*d.* a lb., and gained upon the whole 5*l.*, how much of each did he buy?

Ans. 40 lbs. of tea, and 90 lbs. of coffee.

8. What number is that, which, being severally added to 3, 19, and 51, shall make the results in geometrical progression? *Ans.* 13.

9. It is required to find two geometrical mean proportionals between 3 and 24; and four geometrical means between 3 and 96.

Ans. 6 and 12; and 6, 12, 24, and 48.

10. It is required to find six numbers in geometrical progression, such that their sum shall be 315, and the sum of the two extremes 165.

Ans. 5, 10, 20, 40, 80, and 160.

11. It is required to find the length and breadth of a rectangular field, consisting of two acres of ground, that shall have the same perimeter as a square field consisting of four acres. *Ans.* 43·1868, and 7·4097 poles.

12. After a certain number of men had been employed on a piece of work for 24 days, and had half finished it, 16 men more were set on, by which the remaining half was completed in 16 days: how many men were employed at first; and what was the whole expense, at 1*s.* 6*d.* a day per man? *Ans.* 32 the number of men; and the whole expense 115*l.* 4*s.*

13. It is required to find two numbers, such that if the square of the first be added to the second, the sum shall be 62, and if the square of the second be added to the first, it shall be 176. *Ans.* 7 and 13.

14. The forewheel of a carriage makes six revolutions more than the hind wheel, in going 120 yards; but if the circumference of each wheel was increased by three feet, it would make only four revolutions more than the hind wheel in the same space; what is the circumference of each wheel? *Ans.* 12 and 15 feet.

15. A person bought as many sheep as cost him 98*l.* 16*s.*; one-third of which he sold again at 40*s.* apiece, one-fourth at 36*s.*, and the rest at 34*s.* apiece; and found his gain upon the whole to be 10*l.* 14*s.*; what number of sheep had he? *Ans.* 60.

16. A bankrupt owes A twice as much as he owes B, and C as much as he owes A and B together; now, out of 300*l.*, which is to be divided amongst them, what must each receive? *Ans.* A 100*l.*, B 50*l.*, and C 150*l.*

17. A sum of money is to be divided equally among a certain number of persons; now if there had been 3 claimants less, each would have had 150*l.* more, and if there had been 6 more, each would have had 120*l.* less; required the number of persons, and the sum divided.
Ans. 9 persons, sum 2700*l.*

18. From each of sixteen foreign pieces of gold, of the same denomination, a person filed a fifth of its value, and then offered them all in payment at their nominal currency; but the fraud being detected, and the pieces weighed, they were found to be worth no more than 11*l.* 4*s.*; what was the original value of each piece? *Ans.* 17*s.* 6*d.*

19. A composition of tin and copper, containing 100 cubic inches, was found to weigh 505 ounces; how many ounces of each did it contain, supposing the weight of a cubic inch of copper to be $5\frac{1}{4}$ ounces, and that of a cubic inch of tin $4\frac{1}{4}$ ounces.
Ans. 420 oz. of copper, and 85 oz. of tin.

20. A and B formed a joint stock in trade of 500*l.*, and cleared by the first bargain they made 160*l.*; out of which

A's share came to $32l.$ more than that of B · what sum did each of them advance?

Ans. A $300l.$, and B $200l.$

21. In how many different ways is it possible to pay $100l.$ with seven shilling pieces, and dollars of $4s.\ 6d.$ each?

Ans. 31 different ways.

22. The sum of two numbers is 2, and the sum of their ninth powers is 32; required the numbers by a quadratic equation. *Ans.* $1\pm\sqrt{-1}$ and $1\mp\sqrt{-1}$,

$$\text{or } 1\pm\sqrt{\{-\frac{11}{3}-\frac{2\sqrt{34}}{3}\}} \text{ and } 1\mp\sqrt{\{-\frac{11}{3}-\frac{2\sqrt{34}}{3}\}},$$

$$\text{or } 1\pm\sqrt{\{-\frac{11}{3}+\frac{2\sqrt{34}}{3}\}} \text{ and } 1\mp\sqrt{\{-\frac{11}{3}+\frac{2\sqrt{34}}{3}\}}.$$

23. It is required to find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes. *Ans.* $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5+\sqrt{5})$.

24. The arithmetical mean of two numbers exceeds the geometrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12; what are the numbers?

Ans. 234 and 104.

25. Given $xy(x^2+y^2)=3$, and $x^2y^2(x^4+y^4)=7$, to find the values of x and y.

Ans. $x=\frac{1}{2}(\sqrt{5}+1)$, $y=\frac{1}{2}(\sqrt{5}-1)$.

26. Given $x+y+z=23$, $xy+xz+yz=167$, and $xyz=385$, to find x, y, and z.

Ans. $x=5$, $y=7$, $z=11$.

27. To find four numbers, x, y, z, and w, having the product of every three of them given; viz. $xyz=231$, $xyw=420$, $xzw=660$, and $yzw=1540$.

Ans. $x=3$, $y=7$, $z=11$, and $w=20$.

28. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$, to find the values of x, y, and z.

Ans. $x=10$, $y=17$, and $z=22$.

29. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2+xz=580$, to find the values of x, y, and z.

Ans. $x=9$, $y=3$, and $z=20$.

30. Given $x^2+xy+y^2=5$, and $x^4+x^2y^2+y^4=11$, to find the values of x and y by a quadratic.

Ans. $x=\frac{2}{5}\sqrt{10}+\frac{1}{5}\sqrt{5}$, $y=\frac{2}{5}\sqrt{10}-\frac{1}{5}\sqrt{5}$.

31. Given the equation $x^{4n}-2x^{3n}+x^n=a$, to find the value of x by a quadratic.

Ans. $x=\sqrt[n]{\left\{\frac{1}{2}\pm\sqrt{\left(\frac{3}{4}\pm\sqrt{(\frac{1}{4}+a)}\right)}\right\}}$.

32. It is required to find by what part of the population a people must increase annually, so that they may be doubled at the end of every century.

Ans. By *a* 144*th part nearly.*

33. Required the least number of weights, and the weight of each, that will weigh any number of pounds from 1 to 121 lbs. *Ans.* 1, 3, 9, 27, 81.

34. A person bought as many ducks and geese together as cost him 28*s.*; for the geese he paid 4*s.* 4*d.* apiece, and for the ducks 2*s.* 6*d.* apiece; what number of each had he?

Ans. 3 geese and 6 ducks.

35. It is required to find the least number, which being divided by 6, 5, 4, 3, and 2, shall leave the remainders 5, 4, 3, 2, and 1, respectively. *Ans.* 59.

36. Given the cycle of the sun 18, the golden number or cycle of the moon 8, and the Roman indiction 10, to find the year. *Ans.* 1717.

37. Given $266x-87y=1$, to find the least possible values of x and y in whole numbers.

Ans. $x=52$, and $y=153$.

38. It is required to find two different isosceles triangles, such that their perimeters and areas shall be both expressed by the same numbers.

Ans. Sides of the one 29, 29, 40; and of the other 37, 37, 24.

39. It is required to find the sides of three right angled triangles, in whole numbers, such that their areas shall be all equal to each other.

Ans. 58, 40, 42; 74, 24, 70; 113, 15, 112.

40. Given $x^{\frac{1}{x}} = 1 \cdot 2655$, to find a near approximate value of x. *Ans.* 1·38736.

41. Given $x^y = 5000$, and $y^x = 3000$, to find the values of x and y.

Ans. $x = 4 \cdot 691445$, and $y = 5 \cdot 510132$.

42. Given $x^x + y^y = 285$, and $y^x - x^y = 14$, to find the values of x and y.

Ans. $x = 4 \cdot 016313$, and $y = 2 \cdot 825793$.

43. To find a whole number, such that if unity be added to it, and also to its half, the sums shall be squares.

Ans. 48 or 1680.

44. Required the two least nonquadrate numbers x and y, such that the sum of their squares, and the sum of their cubes, shall be both squares.

Ans. $x = 364$, and $y = 273$.

45. It is required to find two whole numbers, such that their sum shall be a cube, and their product and quotient squares. *Ans.* 25 and 100, or 100 and 900, &c.

46. It is required to find three biquadrate numbers, such that their sum shall be a square.

Ans. 12^4, 15^4, and 20^4.

47. It is required to find three numbers in continued geometrical progression, such that their three differences shall be all squares.

Ans. 567, 1008, and 1792.

48. It is required to find three whole numbers, such that the sum or difference of any two of them shall be square numbers

Ans. 856350, 949986, and 993250.

49. It is required to find two whole numbers, such that their sum shall be a square, and the sum of their squares a biquadrate.

Ans. 4565486027761 and 1061652293520.

50. It is required to find four whole numbers, such that the difference of every two of them shall be a square number.

Ans. 1873432, 2288168, 2399057, and 6560657.

51. It is required to find the sum of the series $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} +$, &c. continued to infinity. *Ans.* $\frac{3}{4}$.

52. It is required to find the sum of the infinite series $\frac{3}{4} - \frac{9}{16} + \frac{27}{64} - \frac{81}{256} + \frac{243}{1024} -$, &c. *Ans.* $\frac{3}{7}$.

53. It is required to find the approximate value of the infinite series of $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} -$, &c.

Ans. ·822467.

54. It is required to find the sum of the series $5 + 6 + 7 + 8 + 9 +$, &c. continued to n terms.

Ans. $\frac{n}{2}(n + 9)$.

55. It is required to find how many figures it would take to express the 25th term of the series $2^1 + 2^2 + 2^4 + 2^8 + 2^{16} +$, &c. *Ans.* 5050446 figures.

56. It is required to find the sum of 100 terms of the series $(1 \times 2) + (3 \times 4) + (5 \times 6) + (7 \times 8) + (9 \times 10) +$, &c.

Ans. 1343300.

57. Required the sum of $1^2 + 2^2 + 3^2 + 4^2 + 5^2 +$, &c. . . $+ 50^2$, which gives the number of shot in a square pile, the side of which is 50. *Ans.* 42925.

58. Required the sum of 25 terms of the series $35 + 36 \times 2 + 37 \times 3 + 38 \times 4 + 39 \times 5$, &c., which gives the number of shot in a complete oblong pile, consisting of 25 tiers, the number of shot in the uppermost row being 35.

Ans. 16575.

THE APPLICATION

OF

ALGEBRA TO GEOMETRY.

In the preceding part of the present performance, I have considered Algebra as an independent science, and con fined myself chiefly to the treating on such of its most useful rules and operations as could be brought within a moderate compass; but as the numerous applications of which it is susceptible ought not to be wholly overlooked, I shall here show, in compliance with the wishes of many respectable teachers, its use in the resolution of geometrical problems; referring the reader to my larger work on this subject, for what relates more immediately to the general doctrine of curves.*

For this purpose it may be observed, that when any proposition of the kind here mentioned is required to be resolved algebraically, it will be necessary, in the first place, to draw a figure that shall represent the several parts or conditions of the problem under consideration, and to regard it as the true one.

Then, having properly considered the nature of the question, the figure so formed must, of necessity, be still further prepared for solution, by producing, or drawing, such lines in it as may appear, by their connexion or rela-

* The learner, before he can obtain a competent knowledge of the method of application above mentioned, must first make himself master of the principal propositions of Euclid, or of those contained in my *Elements of Geometry*; in the latter of which works he will find all the essential principles of the science comprised within a much shorter compass than in the former.

In those cases where it may be requisite to extend this mode of application to trigonometry, mechanics, or any other branch of mathematics, a previous knowledge of the nature and principles of these subjects will be equally necessary.

tions to each other, to be most conducive to the end proposed.

This being done, let the unknown line, or lines, which it is judged will be the easiest to find, together with those that are known, be denoted by the common algebraic symbols, or letters; then, by means of the proper geometrical theorems, make out as many independent equations as there are unknown quantities employed; and the resolution of these, in the usual manner, will give the solution of the problem.

But as no general rules can be laid down for drawing the lines here mentioned, and selecting the properest quantities to substitute for, so as to bring out the most simple conclusions, the best means of obtaining experience in these matters will be to try the solution of the same problem in different ways; and then to apply that which succeeds the best to other cases of the same kind, when they afterwards occur.

The following directions, however, which are extracted, with some alterations, from NEWTON'S *Universal Arithmetic*, and SIMPSON'S *Algebra*, and *Select Exercises*, will often be found of considerable use to the learner, by showing him how to proceed in many cases of this kind, where he would otherwise be left to his own judgment.

1st. In preparing the figure in the manner above mentioned, by producing or drawing certain lines, let them be either parallel or perpendicular to some other lines in it, or be so drawn as to form similar triangles; and, if an angle be given, let the perpendicular be drawn opposite to it, and so as to fall, if possible, from one end of a given line.

2d. In selecting the proper quantities to substitute for, let those be chosen, whether required or not, that are nearest to the known or given parts of the figure, and by means of which the next adjacent parts may be obtained by addition or subtraction only, without using surds.

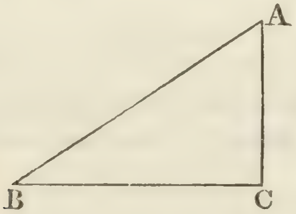
3d. When, in any problem, there are two lines, or quantities, alike related to other parts of the figure or problem, the best way is, not to make use of either of them separately, but to substitute for their sum, difference, or rectangle, or the sum of their alternate quotients, or for some other line or lines in the figure, to which they have both the same relation.

4th. When the area, or the perimeter, of a figure is given, or such parts of it as have only a remote relation to the parts that are to be found, it will sometimes be of use to assume another figure similar to the proposed one that shall have one of its sides equal to unity, or to some other known quantity: as the other parts of the figure, in such cases, may then be determined by the known proportions of their like sides, or parts; and thence the resulting equation required.

These being the most general observations that have hitherto been collected upon this subject, I shall now proceed to elucidate them by proper examples; leaving such further remarks as may arise out of the mode of proceeding here used, to be applied by the learner, as occasion requires, to the solutions of the miscellaneous problems given at the end of the present article.

PROBLEM I.

The base, and the sum of the hypothenuse and perpendicular of a right-angled triangle being given, it is required to determine the triangle.



Let ABC, right angled at C, be the proposed triangle; and put $BC=b$ and $AC=x$.

Then, if the sum of AB and AC be represented by s, the hypothenuse AB will be expressed by $s-x$.

But, by the well-known property of right-angled triangles (Euc. I. 47)*

* The edition of EUCLID, referred to in this and the following problems, is that of Dr. ROBERT SIMPSON, London, 1801; which may also be used in the geometrical construction of these problems, should the student be inclined to exercise his talents upon this elegant, but more difficult branch of the subject.

$$AC^2 + BC^2 = AB^2, \text{ or}$$
$$x^2 + b^2 = s^2 - 2sx + x^2.$$

Whence, omitting x^2, which is common to both sides of the equation, and transposing the other terms, we shall have

$$2sx = s^2 - b^2, \text{ or}$$
$$x = \frac{s^2 - b^2}{2s}$$

which is the value of the perpendicular AC; where s and b may be any numbers whatever, provided s be greater than b.

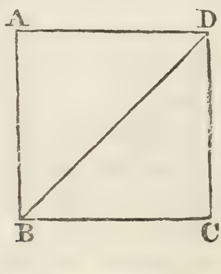
In like manner, if the base and the difference between the hypothenuse and perpendicular be given, we shall have, by putting x for the perpendicular and $d + x$ for the hypothenuse,

$$x^2 + 2dx + d^2 = b^2 + x^2, \text{ or}$$
$$x = \frac{b^2 - d^2}{2d}.$$

Where the base (b) and the given difference (d) may be any numbers as before, provided b be greater than d.

PROBLEM II.

The difference between the diagonal of a square and one of its sides being given, to determine the square.



Let AC be the proposed square, and put the side BC, or CD, $= x$.

Then, if the difference of BD and BC be put $= d$, the hypothenuse BD will be $= x + d$.

But since, as in the former problem, BC^2+CD^2, or $2BC^2=BD^2$, we shall have

$$2x^2=x^2+2dx+d^2, \text{ or}$$
$$x^2-2dx=d^2.$$

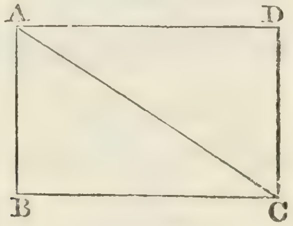
Which equation, being resolved according to the rule laid down for quadratics, in the preceding part of the work, gives

$$x=d+d\sqrt{2}.$$

Which is the value of the side BC, as was required.

PROBLEM III.

The diagonal of a rectangle ABCD, and the perimeter, or sum of all its four sides, being given, to find the sides.



Let the diagonal $AC=d$, half the perimeter $AB+BC=a$, and the base $BC=x$; then will the altitude $AB=a-x$.

And since, as in the former problem, $AB^2+BC^2=AC^2$, we shall have

$$a^2-2ax+x^2+x^2=d^2, \text{ or}$$
$$x^2-ax=\frac{d^2-a^2}{2}.$$

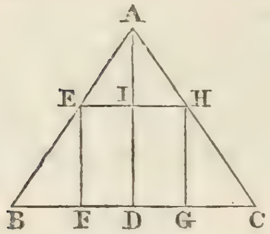
Which last equation, being resolved, gives

$$x=\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{(2d^2-a^2)}.$$

Where a must be taken greater than d and less than $d\sqrt{2}$.

PROBLEM IV.

The base and perpendicular of any plane triangle ABC being given, to find the side of its inscribed square.



Let EG be the inscribed square; and put $BC=b$, $AD=p$, and the side of the square EH or $EF=x$.

Then, because the triangles ABC, AEH, are similar, (Euc. VI. 4) we shall have

$$AD : BC :: AI : EH, \text{ or}$$

$$p : b :: (p-x) : x.$$

Whence, taking the products of the means and extremes, there will arise

$$px=bp-bx.$$

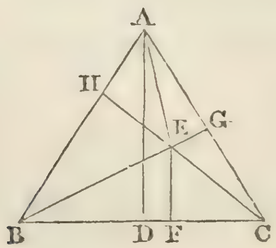
Which, by transposition and division, gives

$$x=\frac{bp}{b+p}.$$

Where b and p may be any numbers whatever, either whole or fractional.

PROBLEM V.

Having the lengths of three perpendiculars, EF, EG, EH, drawn from a certain point E, within an equilateral triangle ABC, to its three sides, to determine the sides.



Draw the perpendicular AD, and having joined EA, EB, and EC, put EF $= a$, EG $= b$, EH $= c$, and BD (which is $\frac{1}{2}$BC) $= x$.

Then, since AB, BC, or CA, are each $= 2x$, we shall have, by Euc. I. 47,

$$\text{AD} = \sqrt{(\text{AB}^2 - \text{BD}^2)} = \sqrt{(4x^2 - x^2)} = \sqrt{3x^2} = x\sqrt{3}.$$

And because the area of any plane triangle is equal to half the rectangle of its base and perpendicular, it follows, that

$$\begin{aligned}
\triangle\,\text{ABC} &= \tfrac{1}{2}\text{BC} \times \text{AD} = x \times x\sqrt{3} = x^2\sqrt{3},\\
\triangle\,\text{BEC} &= \tfrac{1}{2}\text{BC} \times \text{EF} = x \times a \quad = ax,\\
\triangle\,\text{AEC} &= \tfrac{1}{2}\text{AC} \times \text{EG} = x \times b \quad = bx,\\
\triangle\,\text{AEB} &= \tfrac{1}{2}\text{AB} \times \text{EH} = x \times c \quad = cx.
\end{aligned}$$

But the last three triangles, BEC, AEC, AEB, are, together, equal to the whole triangle ABC; whence

$$x^3\sqrt{3} = ax + bx + cx.$$

And, consequently, if each side of this equation be divided by x, we shall have

$$x\sqrt{3} = a + b + c, \text{ or}$$

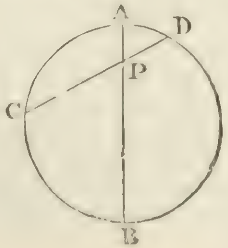
$$x = \frac{a + b + c}{\sqrt{3}}$$

Which is, therefore, half the length of either of the three sides of the triangle.

Cor. Since, from what is above shown, AD is $= x\sqrt{3}$, it follows, that the sum of all the perpendiculars, drawn from any point in an equilateral triangle to each of its sides, is equal to the whole perpendicular of the triangle.

PROBLEM VI.

Through a given point P, in a given circle ACBD, to draw a chord CD, of a given length.



Draw the diameter APB; and put CD$=a$, AP$=b$, PB$=c$, and CP$=x$; then will PD$=a-x$.

But, by the property of the circle, (Euc. III. 35,) CP$\times$PD$=$AP$\times$PB; whence

$$x(a-x)=bc, \text{ or}$$
$$x^2-ax=-bc.$$

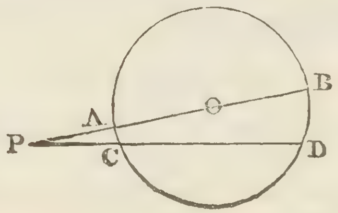
Which equation, being resolved in the usual way, gives

$$x=\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2-bc)};$$

where x has two values, both of which are positive.

PROBLEM VII.

Through a given point P, without a given circle ABDC, to draw a right line so that the part CD, intercepted by the circumference, shall be of a given length.



Draw PAB through the centre O; and put CD$=a$, PA$=b$, PB$=c$, and PC$=x$; then will PD$=x+a$.

But, by the property of the circle, (Euc. III. 36, cor.) PC$\times$PD$=$PA$\times$PB; whence

$$x(x+a)=bc, \text{ or}$$
$$x^2+ax=bc.$$

Which equation being resolved, as in the former problem, gives

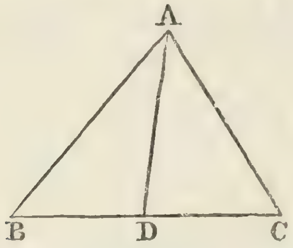
$$x=-\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2+bc)};$$

where one value of x is positive and the other negative.*

* The two last problems, with a few slight alterations, may be readily employed for finding the roots of quadratic equations by construction; but this, as well as the methods frequently given for constructing cubic and some of the higher orders of equations, is a matter of little importance in the present state of mathematical science; analysis, in these cases, being generally thought a more commodious instrument than geometry.

PROBLEM VIII.

The base BC, of any plane triangle ABC, the sum of the sides AB, AC, and the line AD, drawn from the vertex to the middle of the base, being given, to determine the triangle.



Put BD or DC $= a$, AD $= b$, AB + AC $= s$, and AB $= x$; then will AC $= s - x$.

But, by my Geometry, B. II. 19, $\text{AB}^2 + \text{AC}^2 = 2\text{BD}^2 + 2\text{AD}^2$; whence

$$x^2 + (s-x)^2 = 2a^2 + 2b^2, \text{ or}$$
$$x^2 - sx = a^2 + b^2 - \tfrac{1}{2}s^2.$$

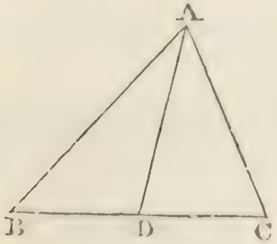
Which last equation, being resolved as in the former instances. gives

$$x = \tfrac{1}{2}s \pm \sqrt{(a^2 + b^2 - \tfrac{1}{4}s^2)},$$

for the values of the two sides AB and AC, of the triangle; taking the sign + for one of them, and − for the other, and observing that $a^2 + b^2$ must be greater than $\frac{1}{4}s^2$.

PROBLEM IX.

The two sides AB, AC, and the line AD, bisecting the vertical angle of any plane triangle ABC, being given, to find the base BC.



Put AB $=a$, AC $=b$, AD $=c$, and BC $=x$; then, by Euc. VI. 3, we shall have

$$\text{AB}(a) : \text{AC}(b) :: \text{BD} : \text{DC}.$$

And consequently, by the composition of ratios, (Euc. v 18,)

$$a+b : a :: x : \text{BD} = \frac{ax}{a+b},$$

and

$$a+b : b :: x : \text{C} = \frac{bx}{a+b}.$$

But, by Euc. VI. B, BD $\times$ DC $+$ AD$^2=$ AB $\times$ AC; wherefore, also,

$$\frac{abx^2}{(a+b)^2}+c^2=ab, \text{ or}$$
$$abx^2=(a+b)^2\times(ab-c^2).$$

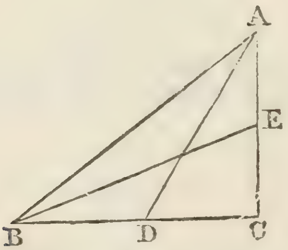
From which last equation we have

$$x=(a+b)\sqrt{\frac{ab-c^2}{ab}},$$

which is the value of the base BC, as required.

PROBLEM X.

Having given the lengths of two lines, AD, BE, drawn from the acute angles of a right-angled triangle ABC, to the middle of the opposite sides, it is required to determine the triangle.



Put AD $=a$, BE $=b$, CD or $\frac{1}{2}$CB $=x$, and CE or $\frac{1}{2}$CA $=y$; then, since (Euc. I. 47) CD$^2+$CA$^2=$AD^2, and CE$^2+$CB$^2=$BE^2, we shall have

$$x^2+4y^2=a^2,$$
$$y^2+4x^2=b^2.$$

Whence, taking the second of these equations from four times the first, there will arise

$$15y^2=4a^2-b^2, \text{ or}$$

$$y=\sqrt{\frac{4a^2-b^2}{15}}.$$

And, in like manner, taking the first of the same equations from four times the second, there will arise

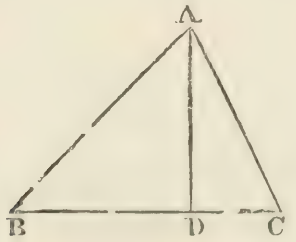
$$15x^2=4b^2-a^2, \text{ or}$$

$$x=\sqrt{\frac{4b^2-a^2}{15}}.$$

Which values of x and y are half the lengths of the base and perpendicular of the triangle; observing that b must be less than $2a$, and greater than $\frac{1}{2}a$.

PROBLEM XI.

Having given the ratio of the two sides of a plane triangle ABC, and the segments of the base, made by a perpendicular falling from the vertical angle, to determine the triangle.



Put BD$=a$, DC$=b$, AB$=x$, AC$=y$, and the ratio of the sides as m to n.

Then, since, by the question, AB : AC :: m : n, and by B. II. 16, of my *Elements of Geometry*, AB$^2-$AC$^2=$BD$^2-$DC^2, we shall have

$$x : y :: m : n, \text{ and}$$

$$x^2-y^2=a^2-b^2$$

But, by the first of these expressions, $nx=my$, or $y=\frac{nx}{m}$; whence, if this be substituted for y in the second, there will arise

$$x^2-\frac{n^2}{m^2}x^2=a^2-b^2, \text{ or}$$

$$(m^2-n^2)x^2=m^2(a^2-b^2)$$

And, consequently, by division and extracting the square root, we shall have

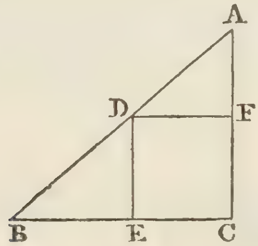
$$x=m\sqrt{\frac{a^2-b^2}{m^2-n^2}}, \text{ and}$$

$$y=n\sqrt{\frac{a^2-b^2}{m^2-n^2}};$$

which are the values of the two sides AB, AC, of the triangle, as was required.

PROBLEM XII.

Given the hypothenuse of a right-angled triangle ABC, and the side DE of its inscribed square, to find the other two sides of the triangle.



Put AB$=h$, DF, or DE$=s$, AC$=x$, and CB$=y$; then, by similar triangles, we shall have

$$\text{AC}(x) : \text{CB}(y) :: \text{AF}(x-s) : \text{FD}(s).$$

And, consequently, by multiplying the means and extremes,

$$xy-sy=sx, \text{ or}$$
$$xy=s(x+y)\ldots.(1).$$

But by Euc. I. 47, $AC^2+CB^2=AB^2$, or

$$x^2+y^2=h^2 \ldots\ldots (2).$$

Whence, if twice equation (1) be added to equation (2), there will arise

$$x^2+2xy+y^2=h^2+2s(x+y), \text{ or}$$
$$(x+y)^2-2s(x+y)=h^2$$

Which equation, being resolved after the manner of a quadratic, gives

$$x+y=s\pm\sqrt{(h^2+s^2)}, \text{ or}$$
$$y=s-x\pm\sqrt{(h^2+s^2)}.$$

Hence, if this value be substituted for y in equation (1), there will arise

$$x\{s-x\pm\sqrt{(h^2+s^2)}\}=s\{s\pm\sqrt{(h^2+s^2)}\}, \text{ or}$$
$$x^2-\{s\pm\sqrt{(h^2+s^2)}\}x=-s\{s\pm\sqrt{(h^2+s^2)}\}.$$

And, consequently, by resolving this last equation, we shall have

$$x=\tfrac{1}{2}\{s\pm\sqrt{(h^2+s^2)}\}\pm\sqrt{\{\tfrac{1}{4}h^2-\tfrac{1}{2}s^2\mp\frac{s}{2}\sqrt{(h^2+s^2)}\}}$$

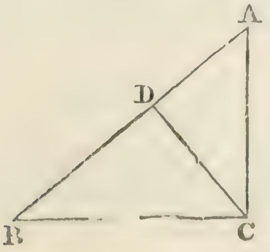
and

$$y=\tfrac{1}{2}\{s\pm\sqrt{(h^2+s^2)}\}\mp\sqrt{\{\tfrac{1}{4}h^2-\tfrac{1}{2}s^2\mp\frac{s}{2}\sqrt{(h^2+s^2)}\}}$$

Which are values of the perpendicular AC and base BC, as was required.

PROBLEM XIII.

Having given the perimeter of a right-angled triangle ABC, and the perpendicular CD, falling from the right angle on the hypothenuse, to determine the triangle.



Put p = perimeter, CD $= a$, AC $= x$, and BC $= y$; then AB$=p-(x+y)$.

But, by Euc. I. 47, AC$^2+$BC$^2=$AB^2; whence

$$x^2+y^2=p^2-2p(x+y)+x^2+2xy+y^2.$$

Or, by transposing the terms and dividing by 2,

$$p(x+y)-\tfrac{1}{2}p^2=xy \,\ldots\ldots\ldots (1).$$

And since, by similar triangles, AB : BC :: AC : CD, we shall also have, by multiplying the means and extremes,

$$AB \times CD = BC \times AC, \text{ or}$$

$$ap-a(x+y)=xy \,\ldots\ldots\ldots (2)$$

Whence, by comparing equation (1) with equation (2), there will arise

$$(a+p)\times(x+y)=ap+\tfrac{1}{2}p^2.$$

$$\therefore x+y=\frac{p(a+\frac{1}{2}p)}{a+p}, \text{ or}$$

$$y=\frac{p(a+\frac{1}{2}p)}{a+p}-x.$$

And if these values be substituted for $x+y$ and y in equation (2), we obtain, by an obvious reduction,

$$(a+p)x^2-p(a+\tfrac{1}{2}p)x=-\tfrac{1}{2}ap^2.$$

From which last equation and the value of y, above found, we shall have

$$x \text{ or AC}=\frac{p(a+\frac{1}{2}p)}{2(a+p)}\pm\frac{p^2}{2(a+p)}\sqrt{\{(a-\tfrac{1}{2}p)^2-2a^2\}}$$

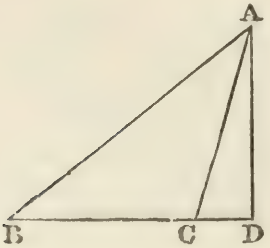
$$y \text{ or BC}=\frac{p(a+\frac{1}{2}p)}{2(a+p)}\mp\frac{p^2}{2(a+p)}\sqrt{\{(a-\tfrac{1}{2}p)^2-2a^2\}},$$

$$\text{and AB}=p-(x+y)=\frac{p^2}{2(a+p)}.$$

Which expressions are, therefore, respectively equal to the values of the three sides of the triangle.

PROBLEM XIV.

Given the perpendicular, base, and sum of the sides of an obtuse-angled plane triangle ABC, to determine the two sides of the triangle



Let the perpendicular AD$=p$, the base BC$=b$, the sum of AB and AC$=s$, and their difference $=x$.

Then, since half the difference of any two quantities added to half their sum gives the greater, and, when subtracted, the less, we shall have

$$\text{AB}=\tfrac{1}{2}(s+x), \text{ and } \text{AC}=\tfrac{1}{2}(s-x).$$

But, by Euc. I. 47, $\text{CD}^2=\text{AC}^2-\text{AD}^2$, or $\text{CD}=\sqrt{\{\frac{1}{4}(s-x)^2-p^2\}}$; and by B. II. 12, $\text{AB}^2=\text{BC}^2+\text{AC}^2+2\text{BC}\times\text{CD}$; whence

$$\tfrac{1}{4}(s+x)^2=b^2+\tfrac{1}{4}(s-x)^2+2b\sqrt{\{\tfrac{1}{4}(s-x)^2-p^2\}}, \text{ or}$$
$$sx-b^2=2b\sqrt{\{\tfrac{1}{4}(s-x)^2-p^2\}}.$$

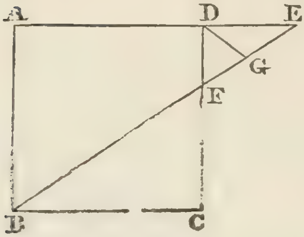
And if each of the sides of this last equation be squared, there will arise, by transposition and simplifying the result,

$$(s^2-b^2)x^2=b^2(s^2-b^2)-4b^2p^2, \text{ or}$$
$$x=b\sqrt{\left(1-\frac{4p^2}{s^2-b^2}\right)}.$$
$$\therefore \text{AB}=\frac{s}{2}+\frac{b}{2}\sqrt{\left(1-\frac{4p^2}{s^2-b^2}\right)},$$
$$\text{and AC}=\frac{s}{2}-\frac{b}{2}\sqrt{\left(1-\frac{4p^2}{s^2-b^2}\right)}.$$

Which are the sides of the triangle, as was required.

PROBLEM XV.

It is required to draw a right line BFE from one of the angles B of a given square BD, so that the part FE, intercepted by DE and DC, shall be of a given length.



Bisect FE in G, and put AB or BC$=a$, FG or GE$=b$, and BG$=x$; then will BE$=x+b$, and BF$=x-b$.

But since, by right-angled triangles, $AE^2=BE^2-AB^2$, we shall have

$$AE=\sqrt{\{(x+b)^2-a^2\}}.$$

And, because the triangles, BCF, EAB, are similar,

$$BF : BC :: BE : AE, \text{ or}$$

$$a(x+b)=(x-b)\sqrt{\{(x+b)^2-a^2\}}.$$

Whence, by squaring each side of the equation, and arranging the terms in order, there will arise

$$x^4-2(a^2+b^2)x^2=b^2(2a^2-b^2).$$

Which equation, being resolved after the manner of a quadratic, will give

$$x=\sqrt{\{a^2+b^2\pm a\sqrt{(a^2+4b^2)}\}},$$

and, consequently, by adding b to, or subtracting it from, this last expression, we shall have

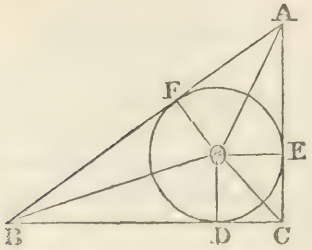
$$BE=\sqrt{\{a^2+b^2\pm a\sqrt{(a^2+4b^2)}\}}+b, \text{ or}$$
$$BF=\sqrt{\{a^2+b^2\pm a\sqrt{(a^2+4b^2)}\}}-b.$$

Which values, by determining the point E or F, will satisfy the problem.

Where it may be observed, that the point G lies in the circumference of a circle, described from the centre D, with the radius FG, or half the given line.

PROBLEM XVI.

The perimeter of a right-angled triangle ABC, and the radius of its inscribed circle being given, to determine the triangle.



Let the perimeter of the triangle $=p$, the radius OD, or OE, of the inscribed circle $=r$, AE$=x$, and BD$=y$.

Then since in the right-angled triangles, AEO, AFO, OE is equal to OF, and AO common, AF will also be equal to AE or x.

And, in like manner, it may be shown, that BF is equal to BD, or y.

But, by the question, and Euc. I. 47, we have

$$(x+r)+(y+r)+(x+y)=p, \text{ and}$$
$$(x+r)^2+(y+r)^2=(x+y)^2.$$

Which respectively give

$$x+y=\tfrac{1}{2}p-r, \text{ and}$$
$$r(x+y)=xy-r^2.$$

From the first of these equations, $y=(\frac{1}{2}p-r)-x$, and if this value be substituted for y in the second, there will arise

$$x^2-(\tfrac{1}{2}p-r)x=-\tfrac{1}{2}pr.$$

Which equation, being resolved in the usual manner gives

$$x=\tfrac{1}{2}(\tfrac{1}{2}p-r)\pm\sqrt{\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr\}},$$

and

$$y=\tfrac{1}{2}(\tfrac{1}{2}p-r)\mp\sqrt{\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr\}}.$$

And, consequently, if r be added to each of these last expressions, we shall have

$$\text{AC}=\tfrac{1}{2}(\tfrac{1}{2}p+r)\pm\sqrt{\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr\}},$$

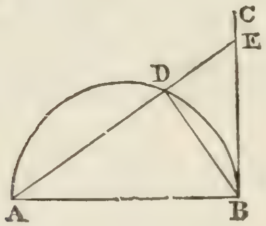
and

$$\text{BC}=\tfrac{1}{2}(\tfrac{1}{2}p+r)\mp\sqrt{\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr\}},$$

for the values of the perpendicular and base of the triangle, as was required.

PROBLEM XVII.

From one of the extremities A, of the diameter of a given semicircle ADB, to draw a right line AE, so that the part DE, intercepted by the circumference and a perpendicular drawn from the other extremity, shall be of a given length.



Let the diameter AB $= d$, DE $= a$, and AE $= x$; and join BD.

Then, because the angle ADB is a right angle, (Euc. III. 31), the triangles ABE, ABD, are similar.

And, consequently, by comparing their like sides, we shall have

$$AE : AB :: AB : AD, \text{ or}$$

$$x : d :: d : x-a.$$

Whence, multiplying the means and extremes of these proportionals, there will arise

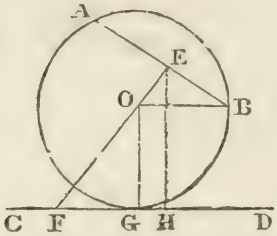
$$x^2 - ax = d^2.$$

Which equation, being resolved after the usual manner, gives

$$x = \tfrac{1}{2}a + \sqrt{(\tfrac{1}{4}a^2 + d^2)}$$

PROBLEM XVIII.

To describe a circle through two given points, A, B, that shall touch a right line CD given in position.



Join AB; and through O, the assumed centre of the required circle, draw FE perpendicular to AB, which will bisect it in E. (Euc. III. 3).

Also, join OB; and draw EH, OG, perpendicular to CD; the latter of which will fall on the point of contact G. (Euc. III. 18).

Hence, since A, E, B, H, F, are given points, put $EB = a$, $EF = b$, $EH = c$, and $EO = x$: then $OF = b - x$.

Because the triangle OEB is right-angled at E, we shall have

$$OB^2 = EO^2 + EB^2, \text{ or}$$

$$OB = \sqrt{(x^2 + a^2)}.$$

But, by similar triangles, FE : EH :: FO : OG or OB.

$$\therefore\ b : c :: b - x : OB$$

$$\therefore\ OB = \frac{c}{b}(b - x).$$

$$\text{Hence, } \sqrt{(x^2 + a^2)} = \frac{c}{b}(b - x).$$

Or, by squaring each side of this equation, and simplifying the result,

$$(b^2 - c^2)x^2 + 2bc^2x = b^2(c^2 - a^2).$$

Which, resolved in the usual manner, gives

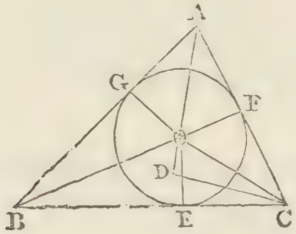
$$x = -\frac{bc^2}{b^2 - c^2} + b\sqrt{\left\{\frac{c^4}{(b^2 - c^2)^2} + \frac{c^2 - a^2}{b^2 - c^2}\right\}},$$

for the distance of the centre O from the chord AB; where b must, evidently, be greater than c.

PROBLEM XIX.

The three lines AO, BO, CO, drawn from the angular

points of a plane triangle ABC, to the centre of its inscribed circle, being given, to find the radius of the circle, and the sides of the triangle.



Let O be the centre of the circle, and, on AO produced, et fall the perpendicular CD; and draw OE, OF, OG, to the points of contact E, F, G.

Then, because the three angles of the triangle ABC are, together, equal to two right angles, (Euc. I. 32) the sum of their halves OAC+OCA+OBE will be equal to one right angle.

But the sum of the two former of these, OAC + OCA, is equal to the external angle DOC: whence the sum of DOC+OBE, as also of DOC+OCD, is equal to a right angle; and, consequently, OBE=OCD.

Let, therefore, AO$=a$, BO$=b$, CO$=c$, and the radius OE, OF or OG$=x$.

Then, since the triangles BOE, COD, are similar, BO : OE :: CO : OD, or $b : x :: c :$ OD; which gives

$$\text{OD}=\frac{cx}{b}, \text{ and } \text{CD}=\sqrt{\left(c^2-\frac{c^2x^2}{b^2}\right)} \text{ or } \frac{c}{b}\sqrt{(b^2-x^2)}.$$

Also, because the triangle AOC is obtuse-angled at O, we shall have, (Euc. II. 12,)

$$\text{AC}^2=\text{AO}^2+\text{CO}^2+2\text{AO}\times\text{OD}; \text{ or}$$

$$\text{AC}=\sqrt{\left(a^2+c^2+\frac{2acx}{b}\right)} \text{ or } \sqrt{\left(\frac{b(a^2+c^2)+2acx}{b}\right)}.$$

But the triangles ACD, AOF, being likewise similar,

$$\text{AC} : \text{CD} :: \text{AO} : \text{OF}, \text{ or}$$

$$\sqrt{\left(\frac{b(a^2+c^2)+2acx}{b}\right)} : \frac{c}{b}\sqrt{(b^2-x^2)} :: a : x.$$

Whence, multiplying the means and extremes, and squaring the result, there will arise

$$bx^2\{b(a^2+c^2)+2acx\}=a^2c^2(b^2-x^2).$$

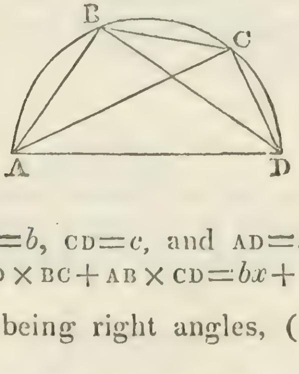
Or, by collecting the terms together, and dividing by the coefficient of the highest power of x,

$$x^3+\left(\frac{ab}{2c}+\frac{ac}{2b}+\frac{bc}{2a}\right)x^2=\frac{abc}{2}.$$

From which last equation x may be determined, and thence the sides of the triangle.*

PROBLEM XX.

Given the three sides AB, BC, CD, of a trapezium ABCD inscribed in a semicircle, to find the diameter, or remaining side AD.



Let AB$=a$, BC$=b$, CD$=c$, and AD$=x$; then, by Euc. VI. D, AC$\times$BD$=$AD$\times$BC$+$AB$\times$CD$=bx+ac$.

But ABD, ACD, being right angles, (Euc. III. 31,) we shall have

$$\text{AC}=\sqrt{(\text{AD}^2-\text{DC}^2)}, \text{ or } \sqrt{(x^2-c^2)}, \text{ and}$$
$$\text{BD}=\sqrt{(\text{AD}^2-\text{AB}^2)}, \text{ or } \sqrt{(x^2-a^2)}.$$

Whence, by substituting these two values in the former expression, there will arise

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* This, and the following problem, cannot be constructed geometrically, or by means only of right lines and a circle, being what the ancients usually denominated solid problems, from the circumstance of their involving an equation of more than two dimensions; in which cases they generally employed the conic sections, or some of the higher orders of curves.

$$\sqrt{(x^2-c^2)} \times \sqrt{(x^2-a^2)} = bx + ac.$$

Or, by squaring each side, and reducing the result,

$$x^3 - (a^2 + b^2 + c^2)x = 2abc$$

From which last equation the value of x may be found, as in the last problem.*

MISCELLANEOUS PROBLEMS.

PROBLEM I.

To find the side of a square, inscribed in a given semi-circle, whose diameter is d. *Ans.* $\frac{1}{5}d\sqrt{5}$.

PROBLEM II.

Having given the hypothenuse (13) of a right-angled triangle, and the difference between the other two sides (7), to find these sides.† *Ans.* 5 and 12.

PROBLEM III.

To find the side of an equilateral triangle, inscribed in a circle whose diameter is d; and that of another circumscribed about the same circle. *Ans.* $\frac{1}{2}d\sqrt{3}$, and $d\sqrt{3}$.

PROBLEM IV.

To find the side of a regular pentagon, inscribed in a circle whose diameter is d. *Ans.* $\frac{1}{4}d\sqrt{(10-2\sqrt{5})}$.

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* Sir I. Newton, in his *Universal Arithmetic*, English edition, 1728, has resolved this problem in a variety of different ways, in order to show that some methods of proceeding, in cases of this kind, frequently lead to more elegant solutions than others; and that a ready knowledge of these can only be obtained by practice.

† Such of these questions as are proposed in numbers, should first be resolved generally, by means of the usual symbols, and then reduced to the answers above given, by substituting the numeral values of the letters in the results thus obtained.

PROBLEM V.

To find the sides of a rectangle, the perimeter of which hall be equal to that of a square, whose side is a, and its rea half that of the square.

Ans. $a+\frac{1}{2}a\sqrt{2}$ and $a-\frac{1}{2}a\sqrt{2}$.

PROBLEM VI.

Having given the side (10) of an equilateral triangle, to nd the radii of its inscribed and circumscribing circles.

Ans. 2·88675 and 5·77350.

PROBLEM VII.

Having given the perimeter (12) of a rhombus, and the um (8) of its two diagonals, to find the diagonals.

Ans. $4+\sqrt{2}$ and $4-\sqrt{2}$.

PROBLEM VIII.

Required the area of a right angled triangle, whose ypothenuse is x^{3x}, and the base and perpendicular x^{2x} nd x^{x}.

Ans. 1·029086.

PROBLEM IX.

Having given the two contiguous sides (a, b) of a paral-logram, and one of its diagonals (d), to find the other iagonal

Ans. $\sqrt{(2a^2+2b^2-d^2)}$.

PROBLEM X.

Having given the perpendicular (300) of a plane triangle, e sum of the two sides (1155), and the difference of the e segments of the base (495), to find the base and the des.

Ans. 945, 375, and 780.

PROBLEM XI

The lengths of the three lines drawn from the three ngles of a plane triangle to the middle of the opposite ides, being 18, 24, and 30, respectively; it is required to nd the sides.

Ans. 20, 28·844, and 34·176.

PROBLEM XII.

In a plane triangle, there is given the base (b), the area (a^2), and the difference of the sides (d) to find the sides and the perpendicular.

Ans. The perp. $p=\dfrac{2a^2}{b}$, and the sides $=$

$$b\sqrt{\left\{\frac{p^2}{b^2-c^2}+\frac{1}{4}\right\}}\pm\frac{c}{2}.$$

PROBLEM XIII.

Given the base (194) of a plane triangle, the line tha bisects the vertical angle (66), and the diameter (200) o the circumscribing circle, to find the other two sides.

Ans. 81·36564, and 157·43952.

PROBLEM XIV.

The lengths of two lines that bisect the acute angles of right-angled plane triangle being a and b, it is required t determine the triangle.

PROBLEM XV.

Given the altitude (4), the base (8), and the sum of th sides (12), of a plane triangle, to find the sides.

Ans. $6+\dfrac{4}{5}\sqrt{5}$ and $6-\dfrac{4}{5}\sqrt{5}$.

PROBLEM XVI.

Having given the base of a plane triangle (15), its are (45), and the ratio of its other two sides as 2 to 3, it is re quired to determine the lengths of these sides.

Ans. 7·79144, and 11·68716.

PROBLEM XVII.

Given the perpendicular (24), the line bisecting the bas (40), and the line bisecting the vertical angle (25), to d termine the triangle. *Ans.* The base $\dfrac{200}{7}\sqrt{14}$.

From which the other two sides may be readily found.

PROBLEM XVIII.

Given the hypothenuse (10) of a right-angled triangle, and the difference of two lines drawn from its extremities to the centre of the inscribed circle (2), to determine the base and perpendicular.

Ans. 8·067442, and 5·909007.

PROBLEM XIX.

Having given the lengths (a, b,) of two chords, cutting each other at right angles, in a circle and the distance (c) of their point of intersection from the centre, to determine the diameter of the circle.

Ans. $\sqrt{\{\frac{1}{2}(a^2 + b^2) + 2c^2\}}$.

PROBLEM XX.

Two trees, standing on a horizontal plane, are 120 feet asunder; the height of the higher is 100 feet, and that of the lower 80; whereabout in the plane must a person place himself, so that his distance from the top of either of the trees shall be equal to the distance between them?

Ans. $20\sqrt{21}$ feet from the bottom of the lower and $40\sqrt{3}$ feet from the bottom of the other.

PROBLEM XXI.

Having given the sides of a trapezium, inscribed in a circle, equal to 6, 4, 5, and 3, respectively, to determine the diameter of the circle.

Ans. $\frac{1}{20}\sqrt{(130 \times 133)}$ or 6·57457.

PROBLEM XXII.

Supposing the town A to be 30 miles from B, B 25 miles from C, and C 20 miles from A; whereabouts must a house be erected that it shall be at an equal distance from each of them? *Ans.* 15·118579 miles from each.

PROBLEM XXIII.

Given the area (100) of an equilateral triangle ABC, whose base BC falls on the diameter, and vertex A in the middle of the arc of a semicircle; required the diameter of the semicircle. *Ans.* $20\sqrt[4]{3}$.

PROBLEM XXIV.

In a plane triangle, having given the perpendicular (p), and the radii (r, R,) of its inscribed and circumscribed circles, to determine the triangle.

$$\textit{Ans.}\text{ The base } \frac{2r\sqrt{(2pR-4rR-r^2)}}{p-2r}.$$

PROBLEM XXV.

Having given the base of a plane triangle equal to $2a$, the perpendicular equal to a, and the sum of the cubes of its other two sides equal to three times the cube of the base; to determine the sides.

$$\textit{Ans. } a\left(2+\frac{1}{3}\sqrt{6}\right) \text{ and } a\left(2-\frac{1}{3}\sqrt{6}\right).$$

ADDENDA.

A New Method of resolving Numerical Equations

As the solution of equations by approximation is one of the most useful, and, at the same time, one of the most tedious operations in modern algebra, several analysts of the first celebrity have turned their attention to this branch of mathematics. LAGRANGE has written a complete work on the subject; and if the method which he has there proposed were as practicable as it is beautiful and complete in theory, nothing further could possibly be desired. But, unfortunately, the number of operations to be performed is so great, that the certainty of the result by no means compensates for the labour of obtaining it, and recourse would always be had to some more expeditious, although less perfect instrument.

Of late, however, two methods have been proposed, nearly at the same time, by MESSRS. HOLROYD and HORNER; which are possessed both of great facility and practical convenience, and are held, on that account, in deserved estimation. Of these, the latter appears decidedly

the most perfect, and is, without doubt, by far the best method of approximation that has hitherto been published; it is, nevertheless, open to two material objections,—the analysis from which it is derived is too high for the subject*, and sufficient provision is not made for determining all the roots in succession.

For these reasons, I have been induced to propose the mode of solving numerical equations that is here treated of; and which possesses the advantage of finding all the roots, whether real or imaginary, by a continuous process; whilst its principles are the same as those commonly employed in the doctrine of algebraic equations.

From the theory of these, laid down in the body of the work, it appears that they are produced by the multiplication of certain simple factors, which, when known, immediately give us all the roots of the equation: it is our present object to discover these factors by a process of division differing but little from that commonly used, and which may be illustrated as follows:

Let the expression

$$\underset{1}{A}x^3+\underset{2}{A}x^2+\underset{3}{A}x+\underset{4}{A}$$

be divided by $x-r$.

* In deriving his rules, Mr. HORNER has unfortunately made use of an analysis much more transcendental than was required; they may be demonstrated from the simplest principles of the differential calculus; but it certainly is to be wished that the rules for approximating to algebraic equations should be demonstrated from antecedent principles; more especially as it is only in the case of these equations that even Mr. HORNER's method is of any use.

Proceeding by the usual method

$$x - r \left\{ A_1 x^3 + A_2 x^2 + A_3 x + A_4 \right\} A_1 x^2 + (A_2 + A_1 r)\, x + (A_3 + A_2 r + A_1 r^2) + \frac{A_4 + A_3 r + A_2 r^2 + A_1 r^3}{x - r}.$$

$$\begin{array}{l}
A_1 x^3 - A_1 r x^2 \\
\hline
\qquad (A_2 + A_1 r) x^2 + A_3 x \\
\\
\qquad (A_2 + A_1 r) x^2 - (A_2 + A_1 r) r x \\
\hline
\qquad\qquad (A_3 + A_2 r + A_1 r^2) x + A_4 \\
\\
\qquad\qquad (A_3 + A_2 r + A_1 r^2) x - (A_3 + A_2 r + A_1 r^2) r \\
\hline
\qquad\qquad\qquad A_4 + A_3 r + A_2 r^2 + A_1 r^3.
\end{array}$$

Where it appears that the several terms in the quotient are performed by adding to the coefficient of the preceding power of x in the dividend, the coefficient of the preceding term in the quotient, multiplied by r. And this law will evidently hold good, however many terms there may be in the dividend.

We might therefore have arranged the division as follows

$$\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \\ 0 & P_1 r & P_2 r & P_3 r \\ \hline P_1 & P_2 & P_3 & P_4 \end{array}$$

Where P_1, P_2, &c., are the several coefficients of the powers of x; the omission of this letter greatly facilitating the process, as will be more distinctly seen in the following example:

Let the expression

$$x^3+4x^2+6x+10$$

be divided by $x-2$.

Arranging the coefficients, and proceeding as above,

$$\begin{array}{cccc} 1 & 4 & 6 & 10 \\ 0 & 2 & 12 & 36 \\ \hline 1 & 6 & 18 & 46 \end{array}$$

And the result is

$$x^2+6x+18+\frac{46}{x-2}$$

To adapt this mode of division to the object we have in view, it will be necessary to modify it, so as to proceed figure by figure, when r contains more than one: and this we may accomplish as follows:

Let r contain two figures, and be represented thus,

$$r=r'+r''.$$

Then since from the preceding operations it appears that any coefficient P_n of the quotient, is equal to

$$A+P_{n-1}.r$$

it follows that

$$P_n=A_n+P_{n-1}.r=A_n+P_{n-1}.(r'+r'')$$
$$=A_n+P_{n-1}.r'+P_{n-1}.r''.$$

And as the two first terms of this result, or $\underset{n}{\mathrm{A}}+\underset{n-1}{\mathrm{P}}.r'$, are the same as before, with the exception of r' being substituted for r; it is evident that the division will commence in the same manner; that is to say, if

$$\underset{1}{\mathrm{A}}x^3+\underset{2}{\mathrm{A}}x^2+\underset{3}{\mathrm{A}}x+\underset{4}{\mathrm{A}}$$

is to be divided by $x-(r'+r'')$, the first part of the operation will stand thus:

$$\begin{array}{cccc}
\underset{1}{\mathrm{A}} & \underset{2}{\mathrm{A}} & \underset{3}{\mathrm{A}} & \underset{4}{\mathrm{A}} \\
0 & \underset{1}{\mathrm{P}}r' & \underset{2}{\mathrm{P}}r' & \underset{3}{\mathrm{P}}r' \\
\hline
\underset{1}{\mathrm{P}} & \underset{2}{\mathrm{P}} & \underset{3}{\mathrm{P}} & \underset{4}{\mathrm{P}}
\end{array}$$

The next step requires a little more attention: it is evident that $\underset{1}{\mathrm{P}}$, $\underset{2}{\mathrm{P}}$, &c. are less than the complete coefficients of the quotient, which we will term $\underset{1}{\mathrm{P}'}$ $\underset{2}{\mathrm{P}'}$, &c., by all that part of the latter that depends on r''; and expressing $\underset{n}{\mathrm{P}'}$, as before, by

$$\underset{n}{\mathrm{P}'}=\underset{n}{\mathrm{A}}+\underset{n-1}{\mathrm{P}'}r'+\underset{n-1}{\mathrm{P}'}.r''$$

it appears that this part of $\underset{n}{\mathrm{P}'}$ may be separated into two others, one $\underset{n-1}{\mathrm{P}'}r''$, which has been wholly neglected; and the other, consisting of that portion $\underset{n-1}{\mathrm{P}'}r'$, which contains r'': now this last is evidently equal to $(\underset{n-1}{\mathrm{P}'}-\underset{n-1}{\mathrm{P}})r'$, for $\underset{n-1}{\mathrm{P}'}$ is the complete quotient, and P is the quotient when r', only, is taken into account; their difference, therefore, must express that part of $\underset{n-1}{\mathrm{P}'}$ which depends on r''.

Whence for the next step in the division, add up every column, as it is found, both without its first term, and with it; multiply the first of these terms by r', and the second by r'', and the results added to the next coefficient in the preceding division, will give the new coefficient sought.

The operation is as follows:

$\underset{1}{A}$	$\underset{2}{A}$	$\underset{3}{A}$	$\underset{4}{A}$
0	$\underset{1}{P}r'$	$\underset{2}{P}r'$	$\underset{3}{P}r'$
$\underset{1}{P}$	$\underset{2}{P}$	$\underset{3}{P}$	$\underset{4}{P}$
0	$\underset{1}{Q}r'$	$\underset{2}{Q}r'$	$\underset{3}{Q}r'$
0	$\underset{1}{P'}r''$	$\underset{2}{P'}r''$	$\underset{3}{P'}r''$
$\underset{1}{Q}$	$\underset{2}{Q}$	$\underset{3}{Q}$	$\underset{4}{Q}$
P'	$\underset{2}{P'}$	$\underset{3}{P'}$	$\underset{4}{P'}$

Where $\underset{1}{Q}$, $\underset{2}{Q}$, &c. are the sums of the columns they stand under, without their first terms $\underset{1}{P}$, $\underset{2}{P}$, &c.

EXAMPLE.

Divide $x^3+3x^2+3x-140$, by $x-4\cdot2$.

1	3		3		−140	
	4		28		124	
1	7		31		−16	
.	0			8	8	96
.		2	1	44	6	648
0		2	2	24	15	608
1	7	2	33	24	−	392

And the result is

$$x^2+7\ 2x+33\cdot24-\frac{\cdot392}{x-4\cdot2}.$$

If the quantity r contains three figures, and is represented by

$$r=r'+r''+r''';$$

we shall have in the same way

A_1	A_2	A_3	A_4
0	$P_1 r'$	$P_2 r'$	$P_3 r''$
P_1	P_2	P_3	P_4
0	$Q_1 r'$	$Q_2 r'$	$Q_3 r'$
0	$P'_1 r''$	$P_2 r''$	$P_3 r''$
Q_1	Q_2	Q_3	Q_4
P'_1	P'_2	P'_3	P'_4
0	$Q'_1 r'$	$Q'_2 r'$	$Q'_3 r'$
0	$Q'_1 r''$	$Q'_2 r''$	$Q'_3 r''$
0	$P''_1 r'''$	$P''_2 r'''$	$P''_3 r''$
Q'	Q'_2	Q'_3	Q'_4
P''_1	P'_2	P''_3	P''_4

To apply what has been here said to the solution of equations, it is only necessary to observe, that if r is a root of the equation, the last column must converge to zero; for this last column is what remains after dividing by $x-r$, and r being a root, $x-r$ is a divisor of the given equation, and, therefore, can leave no remainder.

From which we obtain this practical rule; having determined at what distance from the place of units the first figure of the root stands, substitute in that place every integer successively and divide as above, until the greatest number is found that does not change the sign of the remainder in the last column; set this down as the first figure of the root; and proceed in a similar manner with every figure, until the root is determined with a sufficient accuracy; observing, that whenever nought enters into the result, it will be necessary to examine the preceding figure, in order to determine whether a smaller number would not have left a less remainder.

EXAMPLE I.

Required a root of the equation $x^3 + 60x^2 + 1000x - 1000 = 0$

$r = \cdot 9455148243.$

1	6		1000			−1000	
	0	9	54	81		949	329
	60	9	1054	81		−50	671
		4		36		2	22624
			2	4376		42	291344
	60	94					
		5	2	4736		44	517584
	60	945	1057	2836		−6	153416
		5	·	45			2784825
							123770
	60	9455		2			
						5	2879651
		1		304725			
						5	5788246
	60	94551		309425			
		4824	1057	593025		−	5745914
				45			278505
				2			12376
							1545
				25			
							5288119
				304727			
							5580545
				309452			
						−	0165369
1	60	945514824	1057	623970	2		
				9			5562
				609			244
							30
				618			3
							105762
			1057	624588			
				243			111601
			1057	624831			−53768
				48			2187
			1057	624879			96
							10
				1			1
1	60	945514824	1057	624880			42304
							44598

$$\begin{array}{r} -9170 \\ 432 \\ 16 \\ 2 \\ 8456 \\ \hline 8906 \\ \hline -264 \\ 9 \\ 210 \\ \hline 219 \\ \hline -45 \\ 42 \\ \hline -3 \\ 3 \\ \hline \cdot \end{array}$$

Although the number of figures in this example is much greater than is necessary, as we shall see when we come to speak of the contractions that may be used, yet it affords a sufficient proof of the advantage attending our method; the root is obtained by a process so simple as to be easily remembered, and the result not only gives the root in question, but also the coefficients of the reduced equation, which contains the other two roots; this equation is evidently

$$x^2+60\cdot945514824x+1057\cdot62488=0.$$

In the next example we shall extract all the roots in one continuous operation, and this advantage will then be more distinctly seen.

EXAMPLE II.

Required all the roots of the equation

$$x^3-9x-9=0.$$

Here, it being evident that one root is nearly equal to 1, it will be advantageous to subtract unity from all the roots,

which is accomplished by putting $x=z+1$, the resulting equation is

$$z^3 + 3z^2 - 6z + 1 = 0$$

Whence by the rule

1	3		−6		1	
		1		31	−	569
	3	184791	−5	69		431
				8		2624
				2544	−	434208
				2624		23032
						13456
			−5	4276		10764
				4	−	216565
				32		
				12736	−	19234
						3798
				13456		235
			−5	414144		188
				7		9
				56		−3787
				28		−3355
				22293		443
				2358		30
			−5	411786		24
				9		−487
				7		433
				286		10
				302		
			−5	411484		
				1		
	3	184791	−5	411483		
	1		4	184791		
	4	184791	−1	226692		
		2		2		
				876958		
	4	384791				
		2	1	076958		

Carried forward.

Brought forward.

4	404791
	6
4	410791
	6
4	411391
	8
4	411471
	4
4	411475

149734
2
4
88095
112095
−37639
6
12
12
26464
33784
3855
6
12
12
3
2646
3381
−474
8
16
1
352
449
−25
4
17
21
−4

And the three roots of the reduced equation are

1·184791
1·226684
−4·411475

from which those of the proposed equations may be obtained, by adding the unit that was subtracted.

These examples will sufficiently illustrate the rule for approximating to the real roots of equations; but when the imaginary roots are the objects of research, another and perhaps less commodious method must be employed. In this case the divisor is trinomial, and consequently, two figures are to be determined at each operation instead of one; which leaves a wider scope for ambiguity, and renders the tentative part of the process more fatiguing.

On this account the following method of extracting quadratic divisors may require some further addition; but even in its present state, it is much more practicable and commodious than that given by LAGRANGE; which, as an instrument of calculation, may be considered as almost useless.

The theory is derived, as before, from the common rules of division: for if

$$x^4+\underset{1}{A}x^3+\underset{2}{A}x^2+\underset{3}{A}x+\underset{4}{A}$$

be divided by the quadratic divisor x^2-rx-s the operation will stand as follows.

$$x^2-rx-s\{x^4+\underset{1}{A}x^3+\underset{2}{A}x^2+\underset{3}{A}x+\underset{4}{A}\}\,x^2+(\underset{1}{A}+r)x+\{(\underset{2}{A}+s)+(\underset{1}{A}+r)r\}$$

$$x^4-rx^3-sx^2$$

$$(\underset{1}{A}+r)x^3+(\underset{2}{A}+s)x^2+\underset{3}{A}x$$

$$(\underset{1}{A}+r)x^3-(\underset{1}{A}+r)rx^2-(\underset{1}{A}+r)sx$$

$$\{(\underset{2}{A}+s)+(\underset{1}{A}+r)r\}x^2+\{\underset{3}{A}+(\underset{1}{A}+r)s\}x+\underset{4}{A}$$

$$\{(\underset{2}{A}+s)+(\underset{1}{A}+r)r\}x^2-\{(\underset{2}{A}+s)+(\underset{1}{A}+r)r\}rx-\{(\underset{2}{A}+s)+(\underset{1}{A}+r)r\}s$$

$$\{\underset{3}{A}+(\underset{1}{A}+r)s+(\underset{2}{A}+s)r+(\underset{1}{A}+r)r^2\}x+\{\underset{4}{A}+(\underset{2}{A}+s)s+(\underset{1}{A}+r)rs\}$$

Where it is evident that any coefficient in the quotient, except the last, is obtained, by adding to that of the like power of x in the dividend, the penultimate coefficient in the quotient multiplied by r; and to that of the preceding power of x in the dividend, the antepenultimate coefficient in the quotient multiplied by s: but in the last term, the preceding coefficient multiplied by r, is omitted.

Following this rule, the operation might be arranged thus:

$$\begin{array}{ccccc}
 & A_1 & A_2 & A_3 & A_4 \\
 & P_1 r & P_2 r & P_3 r & P_3 s \\
0 & 0 & P_1 s & P_2 s & \\
\hline
P_1 & P_2 & P_3 & P_4 & P_5
\end{array}$$

When r and s contain more than one figure each, this rule may be modified, so as to proceed figure by figure, in a similar way to that before explained, and the formula will then be as follows:

$$\begin{array}{ccccc}
A_1 & A_2 & A_3 & A_4 & A_5 \\
\cdot & P_1 r' & P_1 s' & P_2 s' & P_3 s' \\
\cdot & \cdot & P_2 r' & P_3 r' & \\
\hline
P_1 & P_2 & P_3 & P_4 & P_5 \\
\cdot & P' r'' & Q_1 r' & Q_1 s' & Q_2 s'' \\
\cdot & \cdot & P_1 s'' & Q_2 r' & P'_3 s' \\
\cdot & \cdot & P'_2 r'' & P'_2 s'' & \cdot \\
\cdot & & \cdot & P'_3 r'' & \cdot \\
\hline
0 & Q_1 & Q_2 & Q_3 & Q_4 \\
\hline
P' & P'_2 & P'_3 & P'_4 & P'_5 \\
\cdot & P'' r''' & Q'_1 r & Q'_1 s' & Q' s \\
 & \cdot & Q' r'' & Q'_2 s'' & Q'_2 s''
\end{array}$$

Carried forward.

Brought forward.

.	.	$\underset{1}{P''}s'''$	$\underset{2}{Q'}r''$	$\underset{3}{P''}s'''$
.	.	$\underset{2}{P''}r'''$	$\underset{2}{Q'}r''$	.
.	.	.	$\underset{2}{P''}s''$	.
.	.	.	$\underset{3}{P''}r'''$	.
$0'$	Q'_1	Q'_2	Q'_3	Q'_4
P''_1	P''_2	P''_3	P''_4	P''_5

The quantities Q_1, Q_2, &c., being, as before, the sums of the columns they stand under, omitting the first terms P_2, P_3, &c.; and the quantities P_1, P_2, P_3, &c., the sums of the same columns including the first terms.

EXAMPLE.

Divide

$$x^4+3x^3+5x^2+7x+9$$

by the quadratic divisor

$$x^2-2\cdot 45x-3\cdot 46$$

1	3		5		7		9	
.	2		3		15		54	
.	.		10		36			
1	5		18		58		63	
		4		8	1	2	10	08
				4	6	72	8	544
			2	16	2	16		
					8	544		
1	5	4	21	36	76	624	81	624
		05		10		15	1	3575
		05		2		2		1810
	5	45		6		905	1	30875
				27·25		181	2	84725
				45·25		327	84	47125
			21	81·25	1	090625		
					2	673625		
					79	297625		

When the remainders in the two last columns vanish, the trinomial is a complete divisor, and consequently, if the dividend is put under the form of an equation, the quadratic divisor will contain two of its roots; whence we have the following rule for finding a trinomial divisor to any given equation.

Determine at what distances from the place of units the first figures in the coefficients of the binomial stand, and set down, in those places, the greatest integers, that, when substituted for r' and s', in the preceding formula, leave the signs of the last columns unchanged. Proceed in a similar way with the next and succeeding figures, until the coefficients are determined with as much accuracy as is required. Observing that when nought enters into either of these numbers, the preceding figure should be examined, in order to try whether the next less digit would not have left a smaller remainder.

EXAMPLE.

Find a trinomial divisor of the equation

$$x^4-36x^2+72x-36=0.$$

$$r=2\cdot 1409$$

$$s=1\cdot 1067$$

0	−36		72		−36	
2	−1		−2		33	
2·1409	4		−66		−3	
	−33		4		−	31
		2	−	1	3	269
	−	1		62	2	959
		21	−	21		
			−3	269	−	41
		31			−	1696
			−2	959		−1696
	−32	69				
		8	1	041	−	22756
		4	−	04		6
		856	−	4		6
				3392		195158
		1696		1696		
	−32	5204	−1	300816		201758
		−6	−	988656		−25802
		−6				−3153
				052344		−315
	−32	5264		-12		−18
		18		−6		22766
		9		−24		
		36		−1284		19280
		−7		−2568		−6522
		1926				
				26664		
		3152		−9		
	−32	523248		−9		
				−5		
				−1498		
				6306		
				315		
				126		
				−29271		
				−25017		
				1647		

Whence the required trinomial is

$$x^2-2\cdot1409x+1\cdot1067.$$

This example sufficiently proves that no great difficulty is to be apprehended in guessing at the successive figures, as the imperfect divisors lead us to the correct integers in nearly every division: thus, to approximate towards the value of r', we divide 72 by 36, the result of which, 2, is correct; to obtain the next figure, we divide 4 by 33, and the result 1 is also correct; and proceeding in the same way with the remaining figures, the required digits are always obtained within a unit of what they should be: to arrive at the first approximation to s', -36 is to be divided by 36, and the result -1, is the number sought; for the second figure, -3, is to be divided by 33, and $-1\cdot09$, the quotient, differs but little from $-\cdot1$, the figure in the coefficient.

It cannot be expected that, in every case, the successive figures will be so readily found, but after a little practice they may be guessed at without much difficulty; more especially if the equation be properly reduced before the operation is commenced.

It has been remarked in a former part of the Addenda, that the process for extracting the real roots is capable of being abridged, and the same remark equally applies to the extraction of the imaginary roots; the nature of this abridgment I shall now explain.

From the formula in page 249, it appears that

$$\underset{3}{P}'=\{\underset{2}{A}+\underset{1}{A}(2r'+r')\}r''$$

$$\underset{3}{P}''=\{\underset{2}{A}+\underset{1}{A}(2r'+2r''+r''')\}r''$$

$$\&c. \qquad\qquad \&c.$$

and consequently, $\underset{3}{P}'$, $\underset{3}{P}''$, &c., may be found by doubling the addition which was made at the preceding step to $\underset{2}{A}$, adding the last figure of the root, and multiplying by that figure: a process the same as that employed to extract the square root.

With this alteration, the example at page 251 would stand as follows.

1	3	−6	
	·1		31
	3·184791	−5	69
			2624
		−5	4276
			13456
		−5	414144
			235
		−5	41179
			28
1	3·184791	−5	41151
	2	4	18479
1	5·184791	−1	22672
	226689	1	07695
	5·411480	−	14977
			11209
			3768
			3378
			−390
			338
			52
			45
			7

1	
	569
	431
	2624
	45724
	434208
	23032
	13456
	10764
	25454
	21656
	3798
	235
	188
	9
	4230
	3787
	443
	28
	22
	493
	486
	7

And the three roots are

$$1{\cdot}184791$$
$$1{\cdot}226689$$
$$-4{\cdot}411480$$

It should be observed that in this example, after finding the root, the resulting quadratic is reduced to another whose roots are less by unity, with a view of facilitating the remaining process. Also, that in the third column, the additions being partly negative, and partly positive, the quantities with unlike signs have been added up separately; but it would be still better, in a case of this kind, to use the arithmetic complements, as is done in logarithms.

It only remains for me to remark that the case in which the usual rules of approximation fail, is when the equation contains two roots that are nearly equal, in which instance the results approximate alternately to these roots; and if their number be even, no unit can be obtained by the usual criterion of the change of sign, since, the parts of the roots between these limits being equal, the results are always positive: and this difficulty will continue until the substitutions be pushed so far as to exhaust the figures which are common to the roots, an operation, in some cases, equivalent to that of resolving the equation.

In an instance of this kind, however, we may always be guided by the convergency of the results, unless the equation contains a pair of imaginary roots whose real parts are nearly equal to the root sought. When this is the case, I am not aware of any method but that of LAGRANGE that can be employed, and it is only in this instance, and then, too, when the equation is of small dimensions, that his beautiful, but impracticable theory can be used with advantage.

ON THE SOLUTION OF EXPONENTIAL EQUATIONS.

IT has been observed in the body of the work, that no direct method exists for resolving the equation

$$x^x = a;$$

and the solution of numerical equations of this kind is there obtained, by applying the rule of double position to the results of successive substitutions.

But this very general process is liable to the same objections in this case, as in that of algebraic equations; the progress of the approximation is not seen, and the operations are not continuous.

Neither of these objections apply to the following very simple mode of solution; which however, unfortunately, does not extend beyond six figures, the range of the common tables of logarithms.

Since

$$x^x = a,$$

taking the logarithms on each side of the equation

$$x . lx = la\,;$$

and repeating the operation

$$lx + l^2x = l^2a,$$

or putting $lx = z$, and $l^2a = a'$

$$z + l.z = a'.$$

Whence we have this rule:

Reduce the equation as above, and seek in the tables the greatest number, which, added to its logarithm, (both being taken to one place only,) will produce a sum less than the first significant figure in a'. Subtract this sum from a' and call the result a''; seek, in like manner, the second figure of the number, which, added to the second figure of its logarithm, does not produce a greater sum than the first significant figure in a''. Subtract this sum from a''; and proceeding as before, we shall at length obtain a number, which, when added to its logarithm, gives a result equal to a'; this number is the logarithm of x.

EXAMPLE.

Required a root of the equation

$$x^x = 100.$$

Here by the given formula

$$z + l.z = \cdot 3010300$$

$$\text{Let} \qquad z = z^1 + z^2 + z^3 + z^4 + \&c.$$

$$l.z = l^1 + l^2 + l^3 + l^4 + \&c.$$

Then in the present case $z' = \cdot 5$, (for $\cdot 6 + \log.(\cdot 6) = \cdot 6 + \bar{1}\cdot 77 = \cdot 37$ is evidently too great,) and proceeding according to the rule,

			$\cdot3010300$
$z^1+l^1=$	$5\ +$	$\bar{1}\cdot6=$	$\cdot1$
			2010300
$z^2+l^2=$	$5\ +$	$14=$	19
			110300
$z^3+l^3=$	$5\ +$	$4=$	9
			20300
$z^4+l^4=$	9	$9=$	18
			2300
$z^5+l^5=$	7	$15=$	22
			100
$z^6+l^6=$	3	3	6
			40

And $z = \cdot555973 = \log. x$, or $x = 3\cdot59728$.

AN

APPENDIX,

CONTAINING A SYNOPSIS ON VARIABLE QUANTITIES.

VARIATION means change of condition or of magnitude; it treats of quantities whose values increase, or decrease, according as we increase, or decrease, the values of other quantities to which they are related.

DEFI. 1. One quantity is said to vary *directly* as another when whatever change is made in the value of the one, the change which takes place in the value of the other is always such that the *ratio* of the two quantities remains the same.

(Ex. 1.) Thus suppose a sum of money is to be divided among a certain number of persons, then the share of each will vary *directly* as the whole sum; for suppose $s =$ sum, $n =$ number of persons, and $p =$ pounds each had, then the original sum must have been each share multiplied by the number of shares, viz. $s = np$, and the ratio of p to np will be $\frac{np}{p} = n$, which remains the same, that is, it is constant, whatever change is made in the value of p. $\therefore p \propto$ *s.

(Ex. 2.) As a second illustration we may remark that in

* It is usual to make this symbol ($\propto$) to denote the word *vary*. Thus,

A varies *directly* as B .... is expressed by $A \propto B$.

A varies *inversely* as B .... by $A \propto \frac{1}{B}$.

A varies as B, and C, *conjointly* .... by $A \propto B \times C$.

A varies *directly* as B and *inversely* as C .... $A \propto \frac{B}{C}$

a series of triangles having the same base,* the areas vary directly as the perpendiculars; for let A be the area of any one of the triangles, B the base, and P the perpendicular, then by mensuration $A = \frac{1}{2}BP$. $\therefore \frac{A}{P} = \frac{\frac{1}{2}BP}{P} = \frac{1}{2}B$, which remains the same, whatever be P.

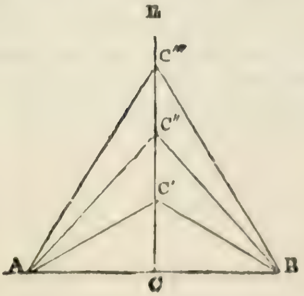
In like manner when the perpendicular is constant, the area varies directly as the base, for $\frac{A}{B} = \frac{\frac{1}{2}BP}{B} = \frac{1}{2}P$, which is by hypothesis constant. $\therefore A \propto B$.

DEFI. 2. One quantity is said to vary *inversely* or *reciprocally* as another when whatever change is made in the value of the one, the change which takes place in the value of the other is always such that the product of the two quantities remains the same.

(Ex. 1.) Thus suppose a sum of money is to be divided among any number of persons, then the share of each will vary *inversely* as the number of persons. For suppose $s =$ given sum and $p =$ the share of each, then n being the number of persons, we must have $n = \frac{s}{p}$, and the product of p and $\frac{s}{p}$ is s, which remains the same, whatever change is made in the value of p. $\therefore p \propto \frac{1}{n}$.

(Ex. 2.) Again in a series of triangles of the same area

* We may conceive this series to be generated thus: let AB be the constant base, and CE any perpendicular to it; then if we suppose C to move along this perpendicular in the direction C′, C″, &c., and join these points to the extremities of the base, the series of triangles will be formed.



the bases vary *inversely* as the perpendiculars.* For calling A the area, B the base, and P the perpendicular, we have $B = \frac{2A}{P}$, and the product of P and $\frac{2A}{P}$ is 2A, which remains constant, however P may vary. $\therefore B \propto \frac{1}{P}$.

DEFI. 3. One quantity (A) is said to vary *conjointly* as *two others*, when it varies directly as their *product*. (See Defi. 1.) And generally one quantity is said to vary *conjointly* as any number of others, when it varies directly as their *product*.

If any factors of this product be constant, then the first (A) varies as the product of the others. For let c represent the product of the constant factors, and P the product of the variable factors,

Then since $A \propto P \times c \; \therefore \frac{A}{Pc} = c'$, a constant quantity. (Defi. 1).

$\therefore \frac{A}{P} = cc'$, a constant quantity. $\therefore A \propto P$.

(Ex. 1.) The simple interest due on money lent varies conjointly as the sum lent, the rate per cent., and the time. For let $I =$ interest, $S =$ sum, $T =$ time, and $R =$ rate, then $I = \frac{S \times R \times T}{100}$, and this divided by the product $S \times R \times T$ gives $\frac{1}{100}$, which remains the same, however S, R, and T vary. $\therefore I \propto S \times R \times T$. Again if any one of the quantities S, R, T, be constant, then I varies conjointly as the other two; and if any two of the quantities S, R, T, are constant, I varies directly as the remaining quantity.

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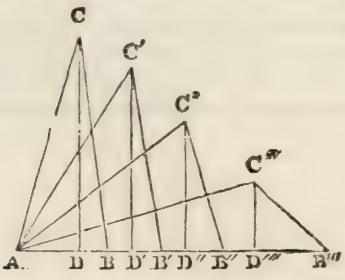
* Such a series of triangles may be exhibited as in the margin, the areas remaining the same while the bases and perpendiculars vary, that is, while they increase or decrease in length.



(Ex. 2.) Again the area of a triangle varies conjointly as the base and perpendicular.* For the product of the base and perpendicular is $B \times P$, and this varies directly as A, or $\frac{1}{2}B \times P$, because $\frac{A}{B \times P} = \frac{\frac{1}{2}B \times P}{B \times P} = \frac{1}{2}$ is constant, whatever change takes place in B or P. $\therefore A \propto BP$.

Defi. 4. One quantity is said to vary *directly* as a second quantity and *inversely* as a third, when the first varies directly as the quotient of the second by the third.

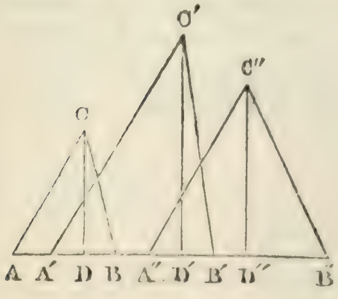
(Ex. 1.) If any sum S be put out at simple interest, the rate per cent. R being constant, then the time T will vary directly as I, the interest, and inversely as the sum. For (Ex. 1 to Defi. 3) $I \propto S \times R \times T$. $\therefore \frac{I}{S \times R} \propto T$, or $T \propto \frac{I}{S \times R}$, in which R is constant.

(Ex. 2.) Again the base of a triangle varies directly as the area, and inversely as the perpendicular. For the area is $\frac{1}{2}B \times P$, and the perpendicular is P, and B varies directly as $\frac{\frac{1}{2}B \times P}{P} = \frac{A}{P} = \frac{1}{2}B$, because their ratio, $\frac{\frac{1}{2}B}{B}$, is constant, viz. $\frac{1}{2}$, whatever be the area and perpendicular. $\therefore B \propto \frac{A}{P}$.

The following theorems are deduced from Mr. Bridge's Algebra, and will further illustrate the doctrine of variation. The manner in which they are established here differs from Mr. Bridge's method, being in accordance with the definitions in the preceding pages.

---

* Thus in any series of triangles, the area of any one ABC, is to the area of any other A″B″C″, as the product AB × CD in the former to the product A″B″ × C″D″ in the latter. We see, therefore, if the base be constant, the area varies as the perpendicular; and if the perpendicular be constant, the area varies as the base (See Ex. 1, Defi. 1), but, as proved above, if neither be constant, the area varies as the product of the two.



Theor. 1. If the first of three quantities vary as the second, and the second vary as the third, then the first shall vary as the third.

That is, if $A \propto B$ and $B \propto C$, then $A \propto C$.

For (Defi. 1) $\frac{A}{B} = c$ and $\frac{B}{C} = c'$, where c and c' stand for constant quantities. $\therefore \frac{A}{B} \times \frac{B}{C} = cc'$ that is $\frac{A}{C} = cc'$ $=$ a constant quantity. $\therefore$ (Defi. 1) A varies as C, or $A \propto C$.

In like manner it may be shown that if there be any number of quantities such that the first varies as the second, the second as the third, and the third as the fourth, and so on, then the first will vary as the last. For let there be four quantities A, B, C, D, (Defi. 1,) $\frac{A}{B} = c$, $\frac{B}{C} = c'$, $\frac{C}{D} = c''$. $\therefore$ $\frac{A \times B \times C}{B \times C \times D} = \frac{A}{D} = cc'c''$, a constant quantity. $\because A \propto D$.

Theor. 2. If the first vary as the second, and the second vary inversely as the third, then shall the first vary inversely as the third.

That is, if $A \propto B$ and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

For (Defi. 1) $\frac{A}{B} = c$ and (Defi. 2) $B \times C = c'$ $\therefore \frac{A}{B} \times B \times C = cc'$ that is, $A \times C = cc' =$ a constant quantity

$$\therefore \text{(Defi. 2) } A \propto \frac{1}{C}.$$

Theor. 3. If the first vary as the third, and the second vary also as the third, then the sum or difference of the first and second shall vary as the third.

That is, if $A \propto C$ and $B \propto C$, then $A \pm B \propto C$.

For (Defi. 1) $\frac{A}{C} = c$ and $\frac{B}{C} = c'$.

$\therefore \frac{A \pm B}{C} = c \pm c' =$ a constant quantity.

$$\therefore \text{(Defi. 1) } A \pm B \propto C$$

Theor. 4. If the first vary as the third, and the second vary as the third also, then shall the third vary as the square root of the product of the first and second.

That is, if $A \propto C$ and $B \propto C$, then $C \propto \sqrt{AB}$.

For (Defi. 1) $\frac{A}{C} = c$ and $\frac{B}{C} = c'$. $\therefore \frac{AB}{C^2} = cc'$. $\therefore \frac{\sqrt{AB}}{C}$ $= \sqrt{cc'} =$ a constant quantity. $\therefore$ (Defi. 1) $C \propto \sqrt{AB}$.

Theor. 5. If the first vary as the second, and the third vary as the fourth, then the product of the first and third shall vary as the product of the second and the fourth.

That is, if $A \propto B$ and $C \propto D$, then $AC \propto BD$.

For (Defi. 1) $\frac{A}{B} = c$ and $\frac{C}{D} = c'$. $\therefore \frac{AC}{BD} = cc' =$ a constant quantity. $\therefore AC \propto BD$.

Cor. If one of these factors is constant, the factor connected with it will vary as the product of the other two

Theor. 6. If the first vary as the second, it shall also vary as any multiple or part of the second.

That is, if $A \propto B$, then $A \propto mB$, where m may be any quantity, either integral or fractional.

For (Defi. 1) $\frac{A}{B} = c$. $\therefore \frac{A}{mB} = \frac{c}{m} =$ a constant quantity. $\therefore A \propto mB$.

Theor. 7. If the first vary as the second, then shall the first, multiplied or divided by any third quantity, vary as the second multiplied or divided by the same quantity.

That is, if $A \propto B$, then $AC \propto BC$, and $\frac{A}{C} \propto \frac{B}{C}$.

For (Defi. 1) $\frac{A}{B} = c \therefore \frac{AC}{BC} = c$, and $\frac{\frac{A}{C}}{\frac{B}{C}} = c \therefore AC \propto BC$, and $\frac{A}{C} \propto \frac{B}{C}$.

It may be further proved that if $A \propto BC$, and $C \propto D$,

then $A \propto BD$. For since $C \propto D \therefore BC \propto BD$, but by hypothesis $A \propto BC$, and $BC \propto BD \therefore$ (Theor. 1) $A \propto BD$.

Also if $A \propto B$, and $C \propto D$, then $AC \propto BD$, and $\frac{A}{C} \propto \frac{B}{D}$;

for $\frac{A}{B} = c$, and $\frac{C}{D} = c' \therefore \frac{AC}{BD} = cc'$, a constant quantity.

$$\therefore AC \propto BD. \text{ Also } \frac{AD}{BC} = \frac{\frac{A}{C}}{\frac{B}{D}} = \frac{c}{c'} = \text{a constant quantity.}$$

$$\therefore \frac{A}{C} \propto \frac{B}{D}.$$

Theor. 8. If the product of two quantities is constant, either must vary inversely as the other.

That is, if $AB = c$, then $A \propto \frac{1}{B}$ and $B \propto \frac{1}{A}$.

For $A \div \frac{1}{B} = AB = c$ and $B \div \frac{1}{A} = AB = c \therefore$ (Defi. 2) $A \propto \frac{1}{B}$ and $B \propto \frac{1}{A}$.

Theor. 9. If one quantity vary as two others conjointly, either of the latter will vary as the first directly and as the other inversely.

That is, if $A \propto BC$, then $B \propto \frac{A}{C}$.

For (Defi. 1) $\frac{A}{BC} = c$, that is $\frac{\frac{A}{C}}{B} = c \therefore$ (Defi. 1) $\frac{A}{C} \propto B$.

Theor. 10. If the first vary as the second, then shall any power or root of the first vary as the same power or root of the second.

That is, if $A \propto B$, then $A^n \propto B^n$.

For (Defi. 1.) $\frac{A}{B} = c \therefore \frac{A^n}{B^n} = c^n =$ a constant quantity $\therefore A^n \propto B^n$.

Theor. 11. If the square of the sum of two quantities vary as the square of their difference, then shall the sum of their squares vary as their product.

That is, if $(A+B)^2 \propto (A-B)^2$, then $(A^2+B^2) \propto AB$.

For (Defi. 1.) $\frac{(A+B)^2}{(A-B)^2}=c$; or $\frac{A^2+B^2+2AB}{A^2+B^2-2AB}=c$ $\therefore$ clearing fractions,

$A^2+B^2+2AB=(A^2+B^2)c-2ABc$; and by transposing,

$(A^2+B^2).(c-1)=2AB(c+1)$ $\therefore \frac{A^2+B^2}{AB}=\frac{2(c+1)}{c-1}=$ a constant quantity. $\therefore$ $(A^2+B^2) \propto AB$.

The foregoing theorems comprehend every thing that can be required in the doctrine of Variation: as a further illustration, however, of this subject, the following additional exercises are given.

(Ex. 1.) If $ax=by$, prove that x varies as y.

$\frac{ax}{by}=c$, multiplying by $\frac{b}{a}$ we have $\frac{x}{y}=\frac{bc}{a}$, which is constant $\because$ $x \propto y$.

(Ex. 2.) If $a^2x=\frac{3y^2}{b}$; show that $x \propto y^2$.

Dividing by a^2, we have $x=\frac{3y^2}{a^2b}$ $\therefore$ $\frac{x}{y^2}=\frac{3}{a^2b}$, which is constant. $\therefore$ $x \propto y^2$.

(Ex. 3.) If $ax+by=cx+dy$, prove that $x \propto y$.

By transposing $(a-c)x=(d-b)y$ $\therefore$ $\frac{x}{y}=\frac{d-b}{a-c}$ which is constant. $\therefore$ $x \propto y$.

(Ex. 4.) Let $x \propto \frac{1}{y}$ and $z \propto \frac{1}{\sqrt{y}}$, to prove that $z^2 \propto x$.

Since $z \propto \frac{1}{\sqrt{y}}$, we know by squaring (Theo. 10) that $z^2 \propto \frac{1}{y}$ $\therefore$ z^2 and x each vary as $\frac{1}{y}$, consequently (Theo. 1) $z^2 \propto x$.

(Ex. 5.) If $x \propto \frac{y^2}{z}$ and $z \propto \frac{1}{\sqrt{x}}$, prove that $y \propto \frac{1}{\sqrt{z}}$.

Defi. 1. $\frac{x}{\frac{y^2}{z}}=c \therefore x=\frac{cy^2}{z}$; substituting this in the second, we have $\frac{z}{\sqrt{\frac{z}{cy^2}}}=c'$ or $y\sqrt{cz}=c' \because y\sqrt{z}=\frac{c'}{\sqrt{c}}$ = a constant quantity.

$$\therefore \text{(Theor. 9)} y \propto \frac{1}{\sqrt{z}}.$$

(Ex. 6.) If $x \propto \frac{1}{y^{\frac{3}{2}}}$ and $z \propto \frac{1}{y}$, show that $x^2 \propto z^3$.

From the second $\frac{z}{\frac{1}{y}}=c \therefore \frac{1}{y}=\frac{z}{c}$; substituting this in the first, $\frac{x}{\frac{z^{\frac{3}{2}}}{c^{\frac{3}{2}}}}=c' \therefore \frac{x}{z^{\frac{3}{2}}}=c^{\frac{3}{2}}c' \therefore$ squaring it will be $\frac{x^2}{z^3}=$ $c^3c'^2=$ a constant quantity.

$$\therefore x^2 \propto z^3.$$

(Ex. 7) If $x \propto \frac{z}{y^2}$, and $z \propto \frac{1}{x^2}$; prove that $x^3 \propto \frac{1}{y^2}$.

From the second $\frac{z}{\frac{1}{x^2}}=c \therefore z=\frac{c}{x^2}$; substituting this in the first, $\frac{x}{\frac{c}{x^2y^2}}=c'$, or $\frac{x^3}{\frac{c}{y^2}}=c'$; multiplying by c, we have $\frac{x^3}{\frac{1}{y^2}}=cc'=$ a constant quantity.

$$\therefore x^3 \propto \frac{1}{y^3}.$$

(Ex. 8.) If $A \propto \frac{B^2C}{D}$ show that $B \propto \sqrt{\frac{AD}{C}}$.

$$\frac{A}{\frac{B^2C}{D}} = c \therefore AD = B^2Cc, \therefore \sqrt{AD} = B\sqrt{Cc}, \therefore \frac{\sqrt{AD}}{B\sqrt{C}} = \sqrt{c};$$

that is, $\sqrt{\frac{AD}{C}} \div B = \sqrt{c}$ a constant quantity

$$\therefore \sqrt{\frac{AD}{C}} \propto B.$$

(Ex. 9.) If the *area* of a triangle varies, show that the base varies as the area *directly*, and the perpendicular altitude of the triangle *inversely*.

Let B = base, P = perpendicular, and A = area.

then $A = \frac{1}{2}BP$ and $\frac{BP}{A} = \frac{B}{\frac{A}{P}} = 2$, which is constant $\therefore B \propto \frac{A}{P}$;

that is, B varies as A directly, and as P inversely

(Ex. 10.) If $\sqrt{A+B} \propto \sqrt{A-B}$, show that $A \propto B$.

$$\frac{\sqrt{A+B}}{\sqrt{A-B}} = c \therefore \frac{A+B}{A-B} = c^2 \therefore A+B = (A-B)c^2$$

$$\therefore (c^2+1)B = (c^2-1)A.$$

$$\therefore \frac{A}{B} = \frac{c^2+1}{c^2-1}, \text{ which is constant } \therefore A \propto B.$$

(Ex. 11.) If x varies *inversely* as y^2, and $ay = \sqrt{bz}$, prove that x varies *inversely* as z.

From the second equation, $y^2 = \frac{bz}{a^2}$, and therefore by the first condition $\frac{bxz}{a^2} = c \quad \therefore xz = \frac{a^2c}{b}$, which is constant.

$$\therefore \text{(Defi. 2.) } x \propto \frac{1}{z}.$$

(Ex. 12.) If x varies *directly* as $y^{\frac{3}{2}}$ and *inversely* as z^2, then supposing z to vary as $\sqrt{x}$, prove that x^4 varies as y^3

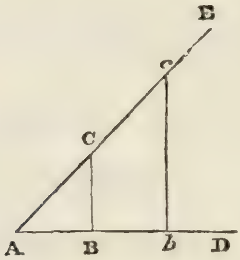
From the first condition $\frac{xz^2}{y^{\frac{3}{2}}}=c$, and from the second

$\frac{z}{\sqrt{x}}=c' \therefore z=c'\sqrt{x}$; substituting this in the first equation, we have $\frac{c'^2x^2}{y^{\frac{3}{2}}}=c \therefore \frac{x^2}{y^{\frac{3}{2}}}=\frac{c}{c'^2}$.

$$\therefore \frac{x^4}{y^3}=\frac{c^2}{c'^4}, \text{ which is constant } \therefore x^4 \propto y^3.$$

(Ex. 13.) Let EAD be half a right angle, and let BC move parallel to itself and perpendicular to AD, then show that AC not only *varies* as BC, but that it is equal to BC$\sqrt{2}$.

(By Euclid, Prop. 47, Book I.) $AB^2+BC^2=AC^2$



But by hypothesis the angles A and C are each half a right angle, and therefore (Euclid, Prop. 5, Book I.) AB=BC,

$$\therefore AC^2=2BC^2=2AB^2 \therefore AC=BC\sqrt{2} \text{ or } AB\sqrt{2};$$

hence $\frac{AC}{BC}=\frac{AC}{AB}=\sqrt{2}$, a constant quantity. $\therefore AC \propto BC$; also $AC \propto AB$.

When variable quantities are mutually dependent, and their actual values in any case known, we may then discover the general relation which subsists among them, as in the following examples.

(Ex. 1.) Let x and y be two variable quantities depending upon each other in such a manner that $x \propto y^2$; and at a certain period of their variation, when $x = a$, y is equal to b. What is the actual relation between x and y?

By the question $\frac{x}{y^2} = c$ and $\frac{a}{b^2} = c$ $\therefore \frac{x}{y^2} = \frac{a}{b^2}$

$\therefore x = \frac{ay^2}{b^2}$ the relation required.

(Ex. 2.) If $(2x+y) \propto (x+5)$; and when $x=2$, $y=5$; what is the equation denoting the relation of x and y?

$\frac{2x+y}{x+5} = c$ and when $x=2$, and $y=5$, then $\frac{4+5}{2+5} = c$

$\therefore \frac{2x+y}{x+5} = \frac{9}{7}$; clearing fractions, $14x+7y=9x+45$

$\therefore 5x = 45-7y$ $\therefore x = \frac{45-7y}{5}$ the relation sought.

(Ex. 3.) Let $x \propto \frac{1}{a+y^2}$; and when $x=b$, $y=c$. Find the equation between x and y.

$\frac{x}{\frac{1}{a+y^2}} = x(a+y^2) = c'$; and when $x=b$ and $y=c$, then

$$b(a+c^2) = c'.$$

$\therefore x(a+y^2) = b(a+c^2)$ $\therefore x = \frac{b(a+c^2)}{a+y^2}$ the required relation.

(Ex. 4.) Let $x+3 \propto \sqrt{y^2+6y}$; and when $x=5$, let $y=2$.

What is the actual equation between x and y?

Since $x=5$ when $y=2$, the first condition gives $\frac{5+3}{\sqrt{4+12}} = c$; that is, $\frac{8}{4} = c = 2$ $\therefore \frac{x+3}{\sqrt{y^2+6y}} = 2$ $\therefore x+3 = 2\sqrt{y^2+6y}$ $\therefore x = 2\sqrt{y^2+6y} - 3$.

(Ex. 5.) Let $ax^2+by^2 \propto cx^2+by+b$; and when $x=1$, let $y=0$. Find the value of x in terms of y.

$\frac{ax^2+by^2}{cx^2+by+b}=c'$; and when $x=1$, and $y=0$, we have

$\frac{a}{c+b}=c'$

$\therefore$ by substitution $\frac{ax^2+by^2}{cx^2+by+b}=\frac{a}{c+b}$ $\therefore$ $(c+b)ax^2+(c+b)by^2=acx^2+aby+ab$ $\therefore$ $abx^2=aby+ab-(c+b)by^2$

$$\therefore x=\sqrt{\frac{aby+ab-(c+b)by^2}{ab}}=\sqrt{y+1-\frac{c+b}{a}.y^2}.$$

(Ex. 6.) If $ax-by^2=cx+dy$, prove that $y \propto \sqrt{x}$.

$\frac{ax-by^2}{cx+dy^2}=c'$ $\therefore$ $ax-by^2=c'cx+c'dy^2$ $\therefore$ $(a-c'c)x=(c'd+b)y^2$

$\therefore$ $\frac{x}{y^2}=\frac{c'd+b}{a-c'c}$ $\therefore$ $\frac{\sqrt{x}}{y}=\sqrt{\frac{c'd+b}{a-c'c}}$, which is constant,

$$\therefore \sqrt{x} \propto y.$$

(Ex. 7.) The quantity v is so related to three other quantities x, y, z, that when y and z are constant, $v \propto x-1$; when x and z are constant, $v \propto y+1$; when x and y are constant, $v \propto z-2$: suppose *all* these quantities to be in a state of *variation*, and in *such manner* that when $v=20$, x, y, and z are respectively equal to 2, 3, and 4; what is the *actual* relation or equatition between v, x, y, and z?

$v \propto x-1$ when y and z are constant.
$v \propto y+1$ when x and z are constant.
$v \propto z-2$ when x and y are constant.

$\therefore$ $\frac{v}{(x-1).(y+1).(z-2)}=c$, let $v=20$, then by the question $x=2$, $y=3$, $z=4$. $\therefore$ $\frac{20}{(2-1).(3+1).(4-2)}=c$

$=\frac{5}{2}$ $\therefore$ $\frac{v}{(x-1).(y+1).(z-2)}=\frac{5}{2}$ $\therefore$ $v=\frac{5}{2}(x-1).(y+1).(z-2)$.

The following theorems, depending upon the first principles of Mechanics, are given as a final illustration.

Let g represent a constant force as gravity, v the velocity of a body influenced by it, t the time of motion, and s the space passed over; then from the first principles of motion we have

$$s=\tfrac{1}{2}tv \text{ and } v=gt.$$

1. Let v be constant and put $\frac{1}{2}v=c$, then $\dfrac{s}{t}=c$;

that is, $s \propto t$ when v is constant.

2 As g is constant, $\dfrac{v}{t}$ is constant; that is, $v \propto t$.

3. Let s be constant, and put $2s=c$ then $tv=c$; that is, $t \propto \dfrac{1}{v}$ or $v \propto \dfrac{1}{t}$, when s is constant.

4. $\dfrac{s}{tv}=\dfrac{1}{2}$ a constant quantity $\therefore$ $s \propto tv$.

5. Let t be constant and put $\frac{1}{2}t=c$, then $\dfrac{s}{v}=c$; that is, $s \propto v$ when t is constant.

Again, let b denote a body moving with the velocity v, and f the moving force constantly acting on it; the body multiplied into its velocity is called the momentum; that is, putting m for the momentum, we have

$m=bv$; also for the velocity, we have $v=\dfrac{ft}{b}$.

1. Let v be constant, then $\dfrac{m}{b}$ is constant; that is, $m \propto b$ when v is constant.

2. Let b be constant, then $\dfrac{m}{v}$ is constant; that is, $m \propto v$, when b is constant.

3. Let m be constant, then bv is constant; that is, $b \propto \dfrac{1}{v}$ or $v \propto \dfrac{1}{b}$.

These all come from the first expression.

4. Also, in the second expression, let b be constant, then

$\frac{v}{ft} = \frac{1}{b}$ a constant quantity; that is, $v \propto ft$, when b is constant.

5. Let v be constant, then $ft \propto b$ because $v = \frac{ft}{b}$.

6. Let t be constant, then $v \div \frac{f}{b}$ is constant.

$\therefore\ v \propto \frac{f}{b}$ when t is constant.

7. Let f be constant, then $v \div \frac{t}{b}$ is constant.

$\therefore\ v \propto \frac{t}{b}$ when f is constant.

THE END.

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A

KEY

TO

BONNYCASTLE'S INTRODUCTION

TO

ALGEBRA;

IN WHICH

THE SOLUTIONS OF ALL THE QUESTIONS, THAT HAVE ONLY THE ANSWERS ANNEXED TO THEM IN THAT WORK, ARE HERE GIVEN AT LENGTH,

IN A MANNER CONFORMABLE TO THE PRESENT IMPROVED STATE OF THE SCIENCE.

BY JOHN BONNYCASTLE,

LATE PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

CORRECTED AND GREATLY IMPROVED

BY SAMUEL MAYNARD.

LONDON:

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EDITOR'S PREFACE:

THE present *Edition* of the Key to Bonnycastle's Algebra is the result of much labour and attention, and may be regarded in a great measure as an entirely new performance. It was the original intention of the Editor to limit himself strictly to the task of revising the reprint of the former impression, with a view to the correction of whatever typographical errors he might be able to detect in going over the analytical processes. But meeting with many gross inaccuracies, he found that, to do justice to the work, he had a more responsible task to perform; he was therefore induced to make considerable alterations, the principal of which will be found in Equations, the Diophantine Analysis, Summation of Series, Miscellaneous Questions, and in the Chapter on the Application of Algebra to Geometry. In these articles the Editor has been very free in suppressing many inelegant and imperfect solutions, and supplying their places with others better adapted to the present improved state of knowledge. Those who may be desirons of ascertaining the extent and estimating the value of those alterations are invited to compare the present work with the corresponding subjects in the preceding Edition.

The Editor, although having been thus burdened with much more labour and anxiety than he anticipated when he first undertook the superintendence of this *Edition*, will nevertheless feel himself amply repaid if it should be found that he has succeeded in rendering any difficulty intelligible to the learner, or in removing a single obstacle that might impede his progress in one of the most valuable and important departments of scientific enquiry.

SAMUEL MAYNARD.

No. 8, *Earl's Court, Leicester Square,*
London; August 18, 1835.

CONTENTS.

KEY

TO

BONNYCASTLE'S ALGEBRA.

Practical Examples for computing the numeral Values of various Algebraic Expressions, or Combinations of Letters.

Required the numeral values of the following quantities; supposing $a=6$, $b=5$, $c=4$, $d=1$, and $e=0$.

1. $2a^2+3bc-5d=72+60-5=127$.

2. $5a^2b-10ab^2+2e=900-1500+0=-600$.

3. $7a^2+b-c\times d+e=252+5-4+0=253$.

4. $5\sqrt{ab}+b^2-2ab-e^2=5\sqrt{30}+25-60-0=$ $(5\times 5\cdot 4772256)+25-60=52\cdot 386128-60=$ $-7\cdot 613872$.

5. $\frac{a}{c}\times d-\frac{a-b}{d}+2a^2e=\left(\frac{6}{4}\times 1\right)-\left(\frac{6-5}{1}\right)+72$ $=\left(\frac{6}{4}-\frac{1}{1}\right)+72=\frac{2}{4}+72=72\frac{1}{2}$.

6. $3\sqrt{c}+2a\sqrt{(2a+b-d)}=3\sqrt{4}+12\sqrt{[12+(5-1)]}$ $=3\sqrt{2^2}+12\sqrt{4^2}=6+48=54$.

7. $a\sqrt{(a^2+b^2)}+3bc\sqrt{(a^2-b^2)}=6\sqrt{(36+25)}+$ $60\sqrt{(36-25)}=6\sqrt{61}+60\sqrt{11}=(6\times 7\cdot 8102497)+$ $(60\times 3\cdot 3166248)=245\cdot 8589862$.

8. $3a^2b+\sqrt[3]{\{c^2+\sqrt{(2ac+c^2)}\}}=540+\sqrt[3]{[16+\sqrt{(48+16)}]}=540+\sqrt[3]{(16+\sqrt{64})}=540+\sqrt[3]{(16+8)}=540+\sqrt[3]{24}=540+2\cdot8844991=542\cdot8844991$.

9. $\frac{2b+c}{3a-c}-\frac{\sqrt{5b}+3\sqrt{c}+d}{2a+c}=\frac{10+4}{18-4}-\frac{\sqrt{25}+3\sqrt{4}+1}{12+4}=\frac{14}{14}-\frac{5+6+1}{16}=1-\frac{12}{16}=1-\frac{3}{4}=\frac{1}{4}$.

ADDITION.

CASE I.

When the quantities are like, and have like signs.

Ex. 4. Ans. $34ay$.
Ex. 5. Ans. $-18by^2$
Ex. 6. Ans. $16a-15x^2$
Ex. 7. Ans. $36ax^2$
Ex. 8. Ans. $16x-17y$
Ex. 9. Ans. $19a+8x^2$

CASE II.

When the quantities are like, but have unlike signs.

Ex. 4. Ans. $+13a^2$
Ex. 5. Ans. $+12ay-6$
Ex. 6. Ans. $+4ab$
Ex. 7. Ans. $-a\sqrt{x}$
Ex. 8. Ans. $-8a^2-2b$
Ex. 9. Ans. $+ax^2+3x^{\frac{1}{2}}$

CASE III.

When some of the quantities are like, and some unlike.

Ex. 4. Ans. $2ax^2$
Ex. 5. Ans. $10a^2x^2-ax+5xy$.
Ex. 6. Ans. $9b^2+3a^2x^2-a^2x+170$.
Ex. 7. Ans. $-5x^2y-3xy^2$.
Ex. 8. Ans. $19x^{\frac{1}{2}}+2\sqrt{xy}+2x^2y+12xy-y+10x$.
Ex. 9. Ans. $37a^2-64x^2-12xy+20-3a\sqrt{x}-2a^{\frac{1}{2}}x^{\frac{1}{2}}$.

EXAMPLES FOR PRACTICE.

Ex. 1.
$$\begin{array}{r} \frac{1}{2}a+\frac{1}{2}b \\ \frac{1}{2}a-\frac{1}{2}b \\ \hline a \quad * \end{array}$$

Ex. 2.
$$\begin{array}{r} 5x-3a+b+7 \\ -3x-4a+2b-9 \\ \hline 2x-7a+3b-2 \end{array}$$

Ex. 3.
$$\begin{array}{r} 2a+3b-4c-9 \\ 5a-3b+2c-10 \\ \hline 7a \qquad -2c-19 \end{array}$$

Ex. 4.
$$\begin{array}{l} 3a+2b-5 \\ a+5b-c \\ 6a-2c+3 \\ \hline 10a+7b-3c-2 \end{array}$$

Ex. 5.
$$\begin{array}{l} 2x^2+ax \\ 3x^2-bx \\ \hline 5x^2+(a-b)x \end{array}$$

Ex. 6.
$$\begin{array}{l} x^3+ax^2+bx+2 \\ x^3+cx^2+dx-1 \\ \hline 2x^3+(a+c)x^2+(b+d)x+1 \end{array}$$

SUBTRACTION.

Ex. 4. Ans. $6xy-30$

Ex. 5. Ans. $9y^2-5y-7$

Ex. 6. Ans. $7-7x-2xy$

Ex. 7. $-8x^2y-8a+7b$

Ex. 8. $\sqrt{ax}-2x^2y+5xy^2$

Ex. 9. $-x^2+2\sqrt{x}+8x-4y$

EXAMPLES FOR PRACTICE.

Ex. 1.
$$\begin{array}{r} \frac{1}{2}a+\frac{1}{2}b \\ \frac{1}{2}a-\frac{1}{2}b \\ \hline +b \end{array}$$

Ex. 2.
$$\begin{array}{r} 3x-2a-b+7 \\ 4x+a-3b+8 \\ \hline -x-3a+2b-1 \end{array}$$

Ex. 3.
$$\begin{array}{r} 3a+b+c-2d \\ -8+b-8c+2d \\ \hline 8+3a+9c-4d \end{array}$$

Ex. 4.
$$\begin{array}{r} 13x^2-2ax+9b^2 \\ 5x^2-7ax-b^2 \\ \hline 8x^2+5ax+10b^2 \end{array}$$

Ex. 5.
$$\begin{array}{r} 20ax-5\sqrt{x}+3a \\ 4ax+5\sqrt{x}-a \\ \hline 16ax-10\sqrt{x}+4a \end{array}$$

Ex. 6.
$$\begin{array}{r} 5ab+2b^2-c+bc-b \\ -2ab+b^2+bc \\ \hline 7ab+b^2-c-b \end{array}$$

Ex. 7.
$$\begin{array}{r} 3x^2+ax+2 \\ 2x^2+bx-4 \\ \hline x^2+(a-b)x+6 \end{array}$$

Ex. 8.
$$\begin{array}{r} ax^3-bx^2+cx-d \\ +bx^2-ex-2d \\ \hline ax^3-2bx^2+(c-e)x+d \end{array}$$

MULTIPLICATION.

CASE I.

When the factors are both simple quantities.

Ex. 5. Ans. $-35a^2bc$

Ex. 6. Ans. $-30a^2x^3$

Ex. 7. Ans $+2x^2y^3$
Ex. 8. Ans. $-42a^2xy^2$
Ex. 9. Ans. $6a^4b^2$
Ex. 10. Ans. $-24a^2x^3y$
Ex. 11. Ans. $-6axy^3z^2$
Ex. 12. Ans. $-2a^2x^2y^3$

CASE II.

When one of the factors is a compound quantity.

Ex. 4. Ans. $60ax-5a^2b$
Ex. 5. Ans. $-70x^3+14ax$
Ex. 6. Ans. $3xy^3+xy^2-2xy$
Ex. 7. Ans. $-26ax^2+2a^3b$
Ex. 8. Ans. $325x^3y+39a^2x^2$
Ex. 9. Ans. $15x^4y-5x^3y^2-10x^2y^3$

CASE III.

When both the factors are compound quantities.

EXAMPLES FOR PRACTICE.

Ex. 1.

$$\begin{array}{llll}
x^2 & -xy & +y^2 & \\
x & +y & & \\
\hline
x^3 & -x^2y & +xy^2 & \\
 & +x^2y & -xy^2 & +y^3 \\
\hline
x^3 & * & * & +y^3 \\
\hline
\end{array}$$

Ex. 2.

$$\begin{array}{lllll}
x^3 & +x^2y & +xy^2 & +y^3 & \\
x & -y & & & \\
\hline
x^4 & +x^3y & +x^2y^2 & +xy^3 & \\
 & -x^3y & -x^2y^2 & -xy^3 & -y^4 \\
\hline
x^4 & * & * & * & -y^4 \\
\hline
\end{array}$$

Ex. 3.

$$\begin{array}{lllll}
x^2 & +xy & +y^2 & & \\
x^2 & -xy & +y^2 & & \\
\hline
x^4 & +x^3y & +x^2y^2 & & \\
 & -x^3y & -x^2y^2 & -xy^3 & \\
 & & +x^2y^2 & +xy^3 & +y^4 \\
\hline
x^4 & * & +x^2y^2 & * & +y^4 \\
\hline
\end{array}$$

Ex. 4.

$$\begin{array}{l}
3x^2-2xy+5\\
\quad x^2+2xy-3\\
\hline
3x^4-2x^3y+5x^2\\
\qquad +6x^3y-4x^2y^2+10xy\\
\qquad\qquad -9x^2+6xy-15\\
\hline
3x^4+4x^3y-4x^2y^2-4x^2+16xy-15
\end{array}$$

Ex. 5.

$$\begin{array}{l}
2a^2-3ax+4x^2\\
5a^2-6ax-2x^2\\
\hline
10a^4-15a^3x+20a^2x^2\\
\qquad -12a^3x+18a^2x^2-24ax^3\\
\qquad\qquad -4a^2x^2+6ax^3-8x^4\\
\hline
10a^4-27a^3x+34a^2x^2-18ax^3-8x^4
\end{array}$$

Ex. 6.

$$\begin{array}{l}
5x^3+4ax^2+3a^2x+a^3\\
2x^2-3ax+a^2\\
\hline
10x^5+8ax^4+6a^2x^3+2a^3x^2\\
\qquad -15ax^4-12a^2x^3-9a^3x^2-3a^4x\\
\qquad\qquad +5a^2x^3+4a^3x^2+3a^4x+a^5\\
\hline
10x^5-7ax^4-a^2x^3-3a^3x^2+a^5
\end{array}$$

Ex. 7.

$$\begin{array}{l}
3x^3+2x^2y^2+3y^3\\
2x^3-3x^2y^2+5y^3\\
\hline
6x^6+4x^5y^2+6x^3y^3\\
\qquad -9x^5y^2-6x^4y^4-9x^2y^5\\
\qquad\qquad +15x^3y^3+10x^2y^5+15y^6\\
\hline
6x^6-5x^5y^2-6x^4y^4+21x^3y^3+x^2y^5+15y^6
\end{array}$$

Ex. 8.

$$\begin{array}{l}
x^3-ax^2+bx-c\\
x^2-dx+e\\
\hline
x^5-ax^4+bx^3-cx^2\\
\quad -dx^4+adx^3-bdx^2+cdx\\
\qquad\qquad +ex^3-aex^2+bex-ce\\
\hline
x^5-(a+d)\,x^4+(b+ad+e)\,x^3-(c+bd+ae)\,x^2\\
\quad +(cd+be)\,x-ce
\end{array}$$

DIVISION.

CASE I.

When the divisor and dividend are both simple quantities.

Ex. 1. $16x^2 \div 8x$, or $\frac{16x^2}{8x} = 2x$; and $\frac{12a^2x^2}{-8a^2x} = -\frac{3x}{2}$

Ex. 2. $-15ay^2 \div 3ay$, or $\frac{-15ay^2}{3ay} = -5y$; and $\frac{-18ax^2y}{-8ax}$ $= 2\frac{1}{4}xy$

Ex. 3. $-\frac{2}{3}a^{\frac{1}{2}} \div \frac{4}{5}a^{\frac{1}{2}} = -\frac{2}{3} \times \frac{5}{4} = -\frac{5}{6}$; and

$ax^{\frac{1}{3}} \div -\frac{3}{5}a^{\frac{1}{2}}x^{\frac{1}{4}} = -\frac{5ax^{\frac{1}{3}}}{3a^{\frac{1}{2}}x^{\frac{1}{4}}} = -\frac{5}{3}a^{\frac{1}{2}}x^{\frac{1}{12}}$

CASE II.

When the divisor is a simple quantity, and the dividend a compound one.

Ex. 1. Here $\frac{3x^3+6x^2+3ax-15x}{3x} = x^2+2x+a-5$

Ex. 2. Here $\frac{3abc+12abx-9a^2b}{3ab} = c+4x-3a.$

Ex. 3. Here $\frac{40a^3b^3+60a^2b^2-17ab}{-ab} = -40a^2b^2$ $-60ab+17$

Ex. 4. Here $\frac{15a^2bc-12acx^2+5ad^2}{-5ac} = -3ab+2\frac{2}{5}x^2$ $-\frac{d^2}{c}$.

Ex. 5. Here $\frac{20ax+15ax^2+10ax+5a}{5a} =$

$\frac{15ax^2+30ax+5a}{5a} = 3x^2+6x+1$

CASE III.

When the divisor and dividend are both compound quantities.

EXAMPLES FOR PRACTICE.

Ex. 1.
$$
\begin{array}{l}
a-x)a^2-2ax+x^2(a-x \\
\qquad\; \underline{a^2-\;ax\qquad} \\
\qquad\qquad -ax+x^2 \\
\qquad\qquad \underline{-ax+x^2}
\end{array}
$$

Ex. 2.
$$
\begin{array}{l}
x-a)x^3-3ax^2+3a^2x-a^3(x^2-2ax+a^2 \\
\qquad\; \underline{x^3+\;ax^2\qquad\qquad} \\
\qquad\qquad -2ax^2+3a^2x \\
\qquad\qquad \underline{-2ax^2+2a^2x\qquad\quad} \\
\qquad\qquad\qquad\qquad\quad a^2x-a^3 \\
\qquad\qquad\qquad\qquad\quad \underline{a^2x-a^3}
\end{array}
$$

Ex. 3.
$$
\begin{array}{l}
a+x)a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2 \\
\qquad\; \underline{a^3+a^2x\qquad\qquad} \\
\qquad\qquad 4a^2x+5ax^2 \\
\qquad\qquad \underline{4a^2x+4ax^2} \\
\qquad\qquad\qquad\quad ax^2+x^3 \\
\qquad\qquad\qquad\quad \underline{ax^2+x^3} \\
\qquad\qquad\qquad\qquad\; * \\
\qquad\qquad\qquad\quad \underline{\qquad\qquad}
\end{array}
$$

Ex. 4.
$$
\begin{array}{l}
y-8)2y^3-19y^2+26y-17(2y^2-3y+2-\dfrac{1}{y-8} \\
\qquad\; \underline{2y^3-16y^2\qquad\quad} \\
\qquad\qquad -3y^2+26y \\
\qquad\qquad \underline{-3y^2+24y\qquad\quad} \\
\qquad\qquad\qquad\qquad 2y-17 \\
\qquad\qquad\qquad\qquad \underline{2y-16} \\
\qquad\qquad\qquad\qquad\quad \underline{-\;\;1}
\end{array}
$$

Ex. 5. $x+1)x^5+1(x^4-x^3+x^2-x+1$

$$\begin{array}{l}
x^5+x^4 \\
\hline
-x^4+1 \\
-x^4-x^3 \\
\hline
x^3+1 \\
x^3+x^2 \\
\hline
-x^2+1 \\
-x^2-x \\
\hline
x+1 \\
x+1 \\
\hline
\end{array}$$

Again. $x-1)x^6-1(x^5+x^4+x^3+x^2+x+1$

$$\begin{array}{l}
x^6-x^5 \\
\hline
x^5-1 \\
x^5-x^4 \\
\hline
x^4-1 \\
x^4-x^3 \\
\hline
x^3-1 \\
x^3-x^2 \\
\hline
x^2-1 \\
x^2-x \\
\hline
x-1 \\
x-1 \\
\hline
\end{array}$$

Ex. 6. $2x-3a)48x^3-76ax^2-64a^2x+105a^3(24x^2-2ax-35a^2$

$$\begin{array}{l}
48x^3-72ax^2 \\
\hline
-4ax^2-64a^2x \\
-4ax^2+6a^2x \\
\hline
-70a^2x+105a^3 \\
-70a^2x+105a^3 \\
\hline
\end{array}$$

Ex. 7. $2x^2+3x-1)4x^4-9x^2+6x-3(2x^2-3x+1$

$$\begin{array}{l}
4x^4+6x^3-2x^2 \\
\hline
-6x^3-7x^2+6x \\
-6x^3-9x^2+3x \\
\hline
2x^2+3x-3 \\
2x^2+3x-1 \\
\hline
-2 \text{ remainder.} \\
\hline
\end{array}$$

Ex. 8.
$$\begin{array}{l}
x^2-ax+a^2)x^4-a^2x^2+2a^3x-a^4(x^2+ax-a^2 \\
\qquad\qquad x^4-a\,x^3+\ a^2x^2 \\
\qquad\qquad \overline{\qquad\qquad ax^3-2a^2x^2+2a^3x} \\
\qquad\qquad\qquad\quad ax^3-\ a^2x^2+\ a^3x \\
\qquad\qquad\qquad\quad \overline{\qquad\quad -a^2x^2+a^3x-a^4} \\
\qquad\qquad\qquad\qquad\quad -a^2x^2+a^3x-a^4 \\
\qquad\qquad\qquad\qquad\quad \overline{\qquad\qquad\qquad}
\end{array}$$

Ex. 9.
$$\begin{array}{l}
3x-6)6x^4-96\quad(2x^3+4x^2+8x+16 \\
\qquad\quad 6x^4-12x^3 \\
\qquad\quad \overline{\qquad\quad 12x^3-96} \\
\qquad\qquad\quad 12x^3-24x^2 \\
\qquad\qquad\quad \overline{\qquad\quad 24x^2-96} \\
\qquad\qquad\qquad\quad 24x^2-48x \\
\qquad\qquad\qquad\quad \overline{\qquad\quad 48x-96} \\
\qquad\qquad\qquad\qquad\quad 48x-96 \\
\qquad\qquad\qquad\qquad\quad \overline{\qquad\qquad}
\end{array}$$

Again.
$$\begin{array}{l}
a+x)a^5+x^5\quad(a^4-a^3x+a^2x^2-ax^3+x^4 \\
\qquad\quad a^5+a^4x \\
\qquad\quad \overline{\qquad -a^4x+x^5} \\
\qquad\qquad -a^4x-a^3x^2 \\
\qquad\qquad \overline{\qquad\qquad a^3x^2+x^5} \\
\qquad\qquad\qquad\quad a^3x^2+a^2x^3 \\
\qquad\qquad\qquad\quad \overline{\qquad -a^2x^3+x^5} \\
\qquad\qquad\qquad\qquad -a^2x^3-ax^4 \\
\qquad\qquad\qquad\qquad \overline{\qquad\qquad ax^4+x^5} \\
\qquad\qquad\qquad\qquad\qquad\quad ax^4+x^5 \\
\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad\quad *\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad\qquad\qquad}
\end{array}$$

Ex. 10. $2x+3)32x^5+243(16x^4-24x^3+36x^2-54x+81$

$$\begin{array}{l}
32x^5+48x^4 \\
\hline
-48x^4+243 \\
-48x^4-72x^3 \\
\hline
72x^3+243 \\
72x^3+108x^2 \\
\hline
-108x^2+243 \\
-108x^2-162x \\
\hline
162x+243 \\
162x+243 \\
\hline
*
\end{array}$$

Again. $x-a)x^6-a^6(x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$

$$\begin{array}{l}
x^6-ax^5 \\
\hline
ax^5-a^6 \\
ax^5-a^2x^4 \\
\hline
a^2x^4-a^6 \\
a^2x^4-a^3x^3 \\
\hline
a^3x^3-a^6 \\
a^3x^3-a^4x^2 \\
\hline
a^4x^2-a^6 \\
a^4x^2-a^5x \\
\hline
a^5x-a^6 \\
a^5x-a^6 \\
\hline
\end{array}$$

Ex. 11. $b-y)b^4-3y^4\ (b^3+b^2y+by^2+y^3$

$$\begin{array}{l}
b^4-b^3y \\
\hline
b^3y-3y^4 \\
b^3y-b^2y^2 \\
\hline
b^2y^2-3y^4 \\
b^2y^2-by^3 \\
\hline
by^3-3y^4 \\
by^3-\ \ y^4 \\
\hline
-2y^4 \text{ remainder.}
\end{array}$$

Again. $a+2b)a^4+4a^2b+16b^4(a^3-2a^2b+4ab+4ab^2-$ [$8b^2+8b^3$

$$
\begin{array}{l}
a^4+2a^3b \\
\hline
-2a^3b+4a^2b \\
-2a^3b-4a^2b^2 \\
\hline
+4a^2b+4a^2b^2 \\
+4a^2b+8ab^2 \\
\hline
+4a^2b^2-\ 8ab^2 \\
+4a^2b^2+\ 8ab^3 \\
\hline
-8ab^2-\ 8ab^3 \\
-8ab^2-16ab^3 \\
\hline
+8ab^3+16b^4 \\
+8ab^3+16b^4 \\
\hline
\end{array}
$$

Ex. 12. $x+a)x^2+px+q(x+(p-a)-\dfrac{ap}{x}+\dfrac{a^2p}{x^2}+\&c.$

$$
\begin{array}{l}
x^2+ax \\
\hline
(p-a)x+q \\
(p-a)x+ap-a^2 \\
\hline
-ap+a^2+q \\
-ap-\dfrac{a^2p}{x} \\
\hline
+\dfrac{a^2p}{x}+a^2+q \\
+\dfrac{a^2p}{x}+\dfrac{a^3p}{x^2} \\
\hline
\&c.\ \&c.
\end{array}
$$

gain. $x-a)x^3-px^2+qx-r(x^2+(a-p)x+(a^2-ap+q)$ $+\&c.$

$$
\begin{array}{l}
x^3-ax^2 \\
\hline
(a-p)x^2+qx \\
(a-p)x^2-(a^2-ap)x \\
\hline
(a^2-ap+q)x \\
(a^2-ap+q)x-a(a^2-ap+q) \\
\hline
+a^3-a^2p+aq \\
\&c.
\end{array}
$$

ALGEBRAIC FRACTIONS.

CASE I.

To find the greatest common measure of the terms of a fraction.

Ex. 4. Here $x^3-a^3)x^4-a^4(x$

$$x^4-a^3x$$

$$a^3x-a^4$$

Divide by a^3, then $x-a)x^3-a^3(x^2+ax+a^2$

$$x^3-ax^2$$

$$ax^2-a^3$$

$$ax^2-a^2x$$

$$a^2x-a^3$$

$$a^2x-a^3$$

Therefore $x-a$ is the greatest common measure sought.

Ex. 5. Here $a^3-a^2x-ax^2+x^3)a^4-x^4 \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (a$

$$a^4-a^3x-a^2x^2+ax^3$$

$$a^3x+a^2x^2-ax^3-x^4$$

Dividing the remainder by x

$$a^3+a^2x-ax^2-x^3)a^3 \quad a^2x-ax^2-x^3(1$$

$$a^3+a^2x-ax^2-x^3$$

$$-2a^2x+2x^3$$

Divide by $2x$; $-a^2+x^2)a^3+a^2x-ax^2-x^3(-a$

$$a^3-ax^2$$

$$a^2x-x^3$$

Divide by x; $a^2-x^2)-a^2+x^2(-1$

$$-a^2+x^2$$

Therefore a^2-x^2 is the greatest common measure sought.

It frequently happens that the common measure of quantities of this kind is better found by resolving both numerator and denominator into their component factors; in which case it will be useful to remember that the difference of any two even powers is divisible both by the difference

and sum of their roots; and that the difference of two odd powers is divisible by the difference of their roots, and their sum by the sum of their roots: Thus, in the last example,

$$\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}=\frac{(a^2+x^2)(a^2-x^2)}{(a^2-x^2)a-(a^2-x^2)x}$$

where it is obvious that a^2-x^2 is a common measure of both terms, and, dividing by it, the fraction reduces to $\frac{a^2+x^2}{a-x}$, which is in its lowest terms: and, consequently, a^2-x^2 is the greatest common measure, as was found by the former rule. And the same method may be advantageously used in other examples of this kind.

EXAMPLE 6.

Here $x^4+a^2x^2+a^4)x^4+ax^3-a^3x-a^4(1$

$$\begin{array}{r} x^4+a^2x^2+a^4 \\ \hline ax^3-a^2x^2-a^3x-2a^4 \end{array}$$

Divide (the remainder above)

by a; $x^3-ax^2-a^2x-2a^3)x^4+a^2x^2+a^4(x+a$

$$\begin{array}{r} x^4-ax^3-a^2x^2-2a^3x \\ \hline ax^3+2a^2x^2+2a^3x+a^4 \\ ax^3-a^2x^2-a^3x-2a^4 \\ \hline 3a^2x^2+3a^3x+3a^4 \end{array}$$

Divide by $3a^2$, and we have

$x^2+ax+a^2)x^3-ax^2-a^2x-2a^3(x-2a$

$$\begin{array}{r} x^3+ax^2+a^2x \\ \hline -2ax^2-2a^2x-2a^3 \\ -2ax^2-2a^2x-2a^3 \\ \hline \end{array}$$

Therefore x^2+ax+a^2 is the greatest common measure sought.

Ex. 7. Multiplying the denominator by 7, we have

$7a^2-23ab+6b^2)35a^3-126a^2b+77ab^2-42b^3(5a-11b$

$$\begin{array}{r} 35a^3-115a^2b+30ab^2 \\ \hline -11a^2b+47ab^2-42b^3 \\ 7 \\ \hline -77a^2b+329ab^2-294b^3 \\ -77a^2b+253ab^2-66b^3 \\ \hline 76ab^2-228b^3 \end{array}$$

Multiply by 7

Dividing by $76b^2$, and we have

$$\begin{array}{l} a-3b)7a^2-23ab+6b^2(7a-2b \\ \quad\quad 7a^2-21ab \\ \hline \quad\quad\quad -2ab+6b^2 \\ \quad\quad\quad -2ab+6b^2 \\ \hline \end{array}$$

Therefore $a-3b$ is the greatest common measure sought.

CASE II.

To reduce Fractions to their lowest terms.

Ex. 4. Here $\dfrac{x^4-a^4}{x^5-a^2x^3}=\dfrac{(x^2+a^2)\ (x^2-a^2)}{x^3(x^2-a^2)}=\dfrac{x^2+a^2}{x^3}$

by dividing both terms by x^2-a^2

Ex. 5. $$\begin{array}{l} 6a^2+7ax-3x^2)6a^2+11ax+3x^2(1 \\ \quad\quad\quad\quad 6a^2+\ 7ax-3x^2 \\ \hline \quad\quad\quad\quad\quad\quad 4ax+6x^2 \end{array}$$

Divide by $2x$: $$\begin{array}{l} 2a+3x)6a^2+7ax-3x^2(3a-x \\ \quad\quad\quad 6a^2+9ax \\ \hline \quad\quad\quad\quad -2ax-3x^2 \\ \quad\quad\quad\quad -2ax-3x^2 \\ \hline \end{array}$$

Therefore $2a+3x)\dfrac{6a^2+\ 7ax-3x^2}{6a^2+11ax+3x^2}=\dfrac{3a-x}{3a+x}$, the fraction sought.

Ex. 6. Here $\dfrac{2x^3-16x-6}{3x^3-24x-9}=\dfrac{2(x^3-8x-3)}{3(x^3-8x-3)}=\dfrac{2}{3}$

Ex. 7. Here, according to the proposed example, we have

$$\frac{9x^5+2x^3+\ 4x^2-x+1}{15x^4-2x^3+10x^2-x+2}$$

Where the numerator, multiplied by 5, gives

$$15x^4-2x^3+10x^2-x+2)45x^5+10x^3+20x^2-5x+5(3x$$
$$45x^5-6x^4+20x^3-3x^2+6x$$
$$\overline{\qquad 6x^4-20x^3+23x^2-11x+5}$$

Mult. by 5

$$15x^4-2x^3+10x^2-x+2)30x^4-100x^3+115x^2-55x+25(2$$
$$30x^4-4x^3+20x^2-2x+4$$
$$\overline{\qquad -96x^3+95x^2-53x+21}$$

Multiply the last divisor by 32, and we shall have

$$-96x^3+95x^2-53x+21)480x^4-64x^3+320x^2-32x+64(-5x$$
$$480x^4-475x^3+265x^2-105x$$
$$\overline{\qquad 411x^3+55x^2-73x+64}$$

Mult. by 32

$$-96x^3+95x^2-53x+21)13152x^3+1760x^2+2336x+2048(137$$
$$13152x^3-13015x^2+7261x-2877$$

Dividing by 4925)$14775x^2-4925x+4925$

$$\overline{\qquad 3x^2-x+1}$$

$$3x^2-x+1)-96x^3+95x^2-53x+21(-32x+21$$
$$-96x^3--32x^2-32x$$
$$\overline{\qquad 63x^2-21x+21}$$
$$63x^2-21x+21$$
$$\overline{\quad * \quad * \quad *}$$

$$\therefore 3x^2-x+1\Big)\frac{9x^5+2x^3+4x^2-x+1}{15x^4-2x^3+10x^2-x+2}\Big(=\frac{3x^3+x^2+1}{5x^2+x+2}=\text{Ans.}$$

CASE III.

To reduce a mixed quantity to an improper fraction.

Ex. 3. Here $1-\dfrac{2x}{a}=\dfrac{1\times a-2x}{a}=\dfrac{a-2x}{a}$ Ans.

Ex. 4. Here $5a-\dfrac{3x-b}{a}=\dfrac{5a^2-(3x-b)}{a}=\dfrac{5a^2-3x+b}{a}$

the fraction required.

Ex. 5. Here $x-\frac{ax+x^2}{2a}=\frac{2ax-(ax+x^2)}{2a}=\frac{ax-x^2}{2a}$

Ex. 6. Here $5+\frac{2x-7}{3x}=\frac{15x+2x-7}{3x}=\frac{17x-7}{3x}$

Ex. 7. Here $1-\frac{x-a-1}{a}=\frac{a-(x-a-1)}{a}=$

$$\frac{a-x+a+1}{a}=\frac{2a-x+1}{a}\text{ Ans.}$$

Ex. 8. Here $1+2x-\frac{x-3}{5x}=\frac{5x(1+2x)-(x-3)}{5a}=$

$$\frac{5x+10x^2-x+3}{5a}=\frac{10x^2+4x+3}{5a}\text{ Ans.}$$

CASE IV.

To reduce an improper fraction to a whole or mixed quantity.

Ex. 2. Here $(ax-x^2)\div x=a-x$. Ans.

Ex. 3. Here $(ab-2a^2)\div ab=(b-2a)\div b=1-\frac{2a}{b}$

Ex. 4. $a-x)a^2+x^2(a+x$

$$\begin{array}{l}
a^2-ax \\
\hline
\quad ax+x^2 \\
\quad ax-x^2 \\
\hline
\qquad 2x^2
\end{array}$$

Therefore $a+x+\frac{2x^2}{a-x}$ is the mixed number sought.

Ex. 5. Here $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$, is as readily found by division.

Ex. 6. Here $(10x^2-5x+3)\div 5x=2x-1+\frac{3}{5x}$ Ans.

CASE V

To reduce fractions to other equivalent ones having a common denominator.

Ex. 2. Here by the rule,

$$\left.\begin{array}{r} 2x\times c=2cx \\ b\times a=ab \end{array}\right\}\text{ new numerators.}$$

$$a\times c=ac \quad \text{common denominator.}$$

Hence $\frac{2cx}{ac}$ and $\frac{ab}{ac}$ are the fractions sought.

Ex. 3. Here

$$\left.\begin{array}{l} a\times c=ac \\ (a+b)b=ab+b^2 \end{array}\right\}\text{ new numerators.}$$

$$b\times c=bc \quad \text{common denominator.}$$

Hence $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$ are the fractions sought.

Ex. 4. Here

$$\left.\begin{array}{r} 3x\times 3c=9cx \\ 2b\times 2a=4ab \\ d\times 2a\times 3c=6acd \end{array}\right\}\text{ new numerators.}$$

$$1\times 3c\times 2a=6ac \quad \text{common denominator.}$$

Hence $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$ are the fractions required.

Ex. 5. Here the three fractions, when reduced, are $\frac{3}{4}$, $\frac{2x}{3}$, and $\frac{5a+4x}{5}$

Therefore

$$\left.\begin{array}{r} 3\times 3\times 5=45 \\ 2x\times 4\times 5=40x \\ (5a+4x)\times 4\times 3=60a+48x \end{array}\right\}\text{ new numerators.}$$

$$4\times 3\times 5=60 \quad \text{common denominator.}$$

The fractions therefore are

$$\frac{45}{60},\ \frac{40x}{60},\ \text{and } \frac{60a+48x}{60}$$

Ex. 6.

$$\text{Here}\left\{\begin{array}{l} a\times 7\times(a-x)=7a^2-7ax \\ 3x\times 2\times(a-x)=6ax-6x^2 \\ 2\times 7\times(a+x)=14a+14x \end{array}\right\}\text{ numerators.}$$

$$2\times 7\times(a-x)=14a-14x \quad \text{com. denom.}$$

Hence $\dfrac{7a^2-7ax}{14a-14x}$, $\dfrac{6ax-6x^2}{14a-14x}$, and $\dfrac{14a+14x}{14a-14x}$, are the fractions required.

CASE VI.

To add fractional quantities together.

Ex. 4. Here $\dfrac{2x}{5}+\dfrac{5x}{7}=\dfrac{14x}{35}+\dfrac{25x}{35}=\dfrac{39x}{35}$ Ans.

Ex. 5. Here $\dfrac{3x}{2a}+\dfrac{x}{5}=\dfrac{15x}{10a}+\dfrac{2ax}{10a}=\dfrac{(15+2a)x}{10a}$ Ans.

Ex. 6. Here $\dfrac{x}{2}+\dfrac{x}{3}+\dfrac{x}{4}=\dfrac{6x}{12}+\dfrac{4x}{12}+\dfrac{3x}{12}=\dfrac{13x}{12}$ Ans.

Ex. 7. Here $\dfrac{4x}{7}+\dfrac{x-2}{5}=\dfrac{20x}{35}+\dfrac{7(x-2)}{35}=\dfrac{20x+7x-14}{35}=$

$$\frac{27x-14}{35} \text{ Ans.}$$

Ex. 8. This may be written $2a+3a+a+\dfrac{2x}{5}-\dfrac{8x}{9}=6a+$

$$\frac{2x}{5}-\frac{8x}{9}=6a+\frac{18x}{45}-\frac{40x}{45}=6a-\frac{22x}{45} \text{ Ans.}$$

Ex. 9. Here the fractions are $\dfrac{3x}{5}$, $\dfrac{a}{a-x}$, and $\dfrac{a-x}{a}$

$$\text{By the rule }\left.\begin{array}{ll} 3x\times(a-x)\times a & =3a^2x-3ax^2 \\ a\times 5\times a & =5a^2 \\ (a-x)\times(a-x)5 & =5a^2-10ax+5x^2 \end{array}\right\}\text{numerators.}$$

Their sum $=10a^2-10ax+3a^2x-3ax^2+5x^2$, and $5\times(a-x)\times a=5a^2-5ax$ common denominator.

Hence the sum is $2a+\dfrac{10a^2-10ax+3a^2x-3ax^2+5x^2}{5a^2-5ax}$

Ex. 10. This is the same as $9x+\frac{x-2}{3}-\frac{2x-3}{5x}=9x+\frac{5x^2-10x}{15x}-\frac{6x-9}{15x}=9x+\frac{5x^2-10x-6x+9}{15x}=9x+\frac{5x^2-16x+9}{15x}$ Ans.

Ex. 11. Here $5x+\frac{2a}{3x^2}+\frac{a+2x}{4x}=5x+\frac{8ax}{12x^3}+\frac{3ax^2+6x^3}{12x^3}=5x+\frac{8ax+3ax^2+6x^3}{12x^3}$ Ans.

We have not in all these examples followed exactly the process described in the rule, at least not so as to exhibit the operations, both in order to save room, and to indicate to the student a more concise method of setting down his work; and the same is also observed in the following case.

CASE VII.

To subtract one fractional quantity from another.

Ex. 3. Here $\frac{12x}{7}-\frac{3x}{5}=\frac{60x}{35}-\frac{21x}{35}=\frac{39x}{35}$ Ans.

Ex. 4. Here $15y-\frac{1+2y}{8}=\frac{120y-(1+2y)}{8}=\frac{118y-1}{8}$

Ex. 5. Here $\left.\begin{array}{l} ax\times(b+c)=abx+acx \\ ax\times(b-c)=abx-acx \end{array}\right\}$ numerators.

The difference $=$ $2acx$

Then $(b-c)\times(b+c)=b^2-c^2$ common denominator.

Hence $\frac{2acx}{b^2-c^2}$ is the difference sought.

Ex. 6. Here $x+\frac{x}{2b}-(x-\frac{x-a}{c})=\frac{x}{2b}+\frac{x-a}{c}=$

$$\frac{cx}{2bc}+\frac{2bx-2ba}{2bc}=\frac{cx+2bx-2ba}{2bc} \text{ Ans.}$$

Ex. 7. Here again we have $a+\frac{a-x}{a+x}-(a-\frac{a+x}{a-x})=$

$$\frac{a-x}{a+x}+\frac{a+x}{a-x}$$

Whence, by the preceding rule,

$$\left.\begin{array}{l}(a-x)\times(a-x)=a^2-2ax+x^2\\(a+x)\times(a+x)=a^2+2ax+x^2\end{array}\right\}\text{ numerators.}$$

$$\text{Sum }=2a^2 \quad * \quad +2x^2$$

$(a+x)\times(a-x)=a^2-x^2$ common denominator.

Therefore $\frac{2a^2+2x^2}{a^2-x^2}$ is the difference required.

Ex. 8. This is the same, when properly arranged, as

$$ax-x+\frac{2x+7}{8}+\frac{5x-6}{21}=ax-x+\frac{42x+147}{168}+\frac{40x-48}{168}=$$

$$ax-x+\frac{82x+99}{168}=\frac{168ax-168x+82x+99}{168}$$

$$=\frac{168ax-86x+99}{168} \text{ the answer required.}$$

Ex. 9. Here subtracting the first from the second we have $x+\frac{11x-10}{15}-\frac{3x-5}{7}=\frac{26x-10}{15}-\frac{3x-5}{7}=$

$\frac{182x-70}{105}-\frac{45x-75}{105}=\frac{137x+5}{105}$ the answer sought.

Ex. 10. First to find the difference of the fractions

$$\frac{a-x}{a(a+x)} \text{ and } \frac{a+x}{a(a-x)}$$

$$\left.\begin{array}{l}a(a-x)\times(a-x)=a^3-2a^2x+ax^2\\a(a+x)\times(a+x)=a^3+2a^2x+ax^2\end{array}\right\}\text{ numerators.}$$

$$\text{Difference }=-4a^2x$$

$a(a+x)\times a(a-x)=a^4-a^2x^2$ common denominator.

Hence the second fraction subtracted from the first is

$$\frac{-4a^2x}{a^4-a^2x^2}=\frac{-4x}{a^2-x^2};$$

and consequently $a-\frac{4x}{a^2-x^2}$ is the difference sought.

CASE VIII.

To multiply fractional quantities together.

Ex. 4. Here $\frac{\dot{3}x}{2}$*$\times\frac{5x}{\dot{3}b}=\frac{5x^2}{2b}$ Ans.

Ex. 5. Here $\frac{\dot{2}x}{5}\times\frac{3x^2}{\dot{2}a}=\frac{3x^2}{5a}$ Ans.

Ex. 6. Here $\frac{2x}{3}\times\frac{4x^2}{7}\times\frac{a}{a+x}=\frac{8x^3a}{21a+21x}$ Ans.

Ex. 7. Here $\frac{\dot{2}x}{\dot{a}}\times\frac{3\dot{a}\dot{b}}{\dot{c}}\times\frac{5a\dot{c}}{\dot{2}\dot{b}}=15ax$ Ans.

Ex. 8. By reducing these to improper fractions they become $\frac{2a^2+bx}{a}\times\frac{6a^2x-b}{ax}=\frac{12a^4x+6a^2bx^2-2a^2b-b^2x}{a^2x}$

It is, however, frequently more simple to multiply quantities of this kind together, as in common multiplication: thus,

$$\begin{array}{r}
2a+\dfrac{bx}{a} \\
6a-\dfrac{b}{ax} \\
\hline
12a^2+6bx \\
-\dfrac{2b}{x}-\dfrac{b^2}{a^2} \\
\hline
12a^2+6bx-\dfrac{2b}{x}-\dfrac{b^2}{a^2}
\end{array}$$ the product,

which is equivalent to the preceding fraction.

Ex. 9. Here $\frac{3x}{1}\times\frac{x+1}{2a}\times\frac{x-1}{a+b}=\frac{3x^3-3x}{2a^2+2ab}$ Ans.

* These points are placed here to denote such factors as cancel each other.

Ex. 10. The third fraction $a+\frac{ax}{a-x}=\frac{a^2}{a}$.

So that we have $\frac{a^2-x^2}{a+b}\times\frac{a^2-b^2}{ax+x^2}\times\frac{a^2}{a-x}$

Now observing what has been before stated relative to the factors $a^2-x^2=(a+x)(a-x)$, and $a^2-b^2=(a+b)(a-b)$, also $ax+x^2=x(a+x)$, our product may be written in the form

$$\frac{(a+x)(a-x)(a+b)(a-b)\times a^2}{(a+b)(a+x)x(a-x)}$$

Which, by cancelling such factors in the numerator and denominator as are alike, becomes

$$\frac{(a-b)a^2}{x}=\frac{a^3-a^2b}{x}, \text{ the answer.}$$

CASE IX.

To divide one fractional quantity by another.

Ex. 5. Here $\frac{7x}{5}\div\frac{3}{x}=\frac{7x}{5}\times\frac{x}{3}=\frac{7x^2}{15}$ Ans.

Ex. 6. Here $\frac{4x^2}{7}\div\frac{5x}{1}=\frac{4x^2}{7}\times\frac{1}{5x}=\frac{4x}{35}$ Ans.

Ex. 7. Here $\frac{x+1}{6}\times\frac{3}{2x}=\frac{x+1}{4x}$ Ans.

Ex. 8. Here $\frac{x}{1-x}\times\frac{5}{x}=\frac{5}{1-x}$ Ans.

Ex. 9. This expression here may be written

$$\frac{(2a+x)x}{c^3-x^3}\times\frac{c-x}{x}=\frac{(2a+x)x}{(c^2+cx+x^2)(c-x)}\times\frac{c-x}{x}$$

$$=\frac{2a+x}{c^2+cx+x^2} \text{ Ans.}$$

Where it is to be observed that $c-x$ is a divisor of c^3-x^3 as stated in Case I.

Ex. 10. These two fractions are readily resolved into the following factors:

$$\frac{(x^2+b^2)(x^2-b^2)}{(x-b)(x-b)}\times\frac{x-b}{x(x+b)}=\frac{(x^2+b^2)(x^2-b^2)}{x(x^2-b^2)}=$$
$$\frac{x^2+b^2}{x}\text{ Ans.}$$

INVOLUTION.

RULE I.

Ex. 1. $(2a^2)^3=2^3a^6=8a^6$ Ans.

Ex. 2. $(2a^2x)^4=2^4a^8x^4=16a^8x^4$ Ans.

Ex. 3. $(-\frac{2}{3}x^2y^3)^3=\left(-\frac{2}{3}\right)^3x^6y^9=-\frac{8}{27}x^6y^9$ Ans.

Ex. 4. $\left(-\frac{3a^2x}{5b^2x}\right)^4=+\frac{3^4a^8x^4}{5^4b^8x^4}=\frac{81a^8x^4}{625b^8x^4}$

Ex. 5. In the preceding page of the Introduction, we have

$$\begin{array}{l}
(a+x)^3=a^3+3a^2x+3ax^2+x^3\\
\text{Mult. by } a+x\\
\hline
\quad a^4+3a^3x+3a^2x^2+\ \ ax^3\\
\qquad\ \ +\ \ a^3x+3a^2x^2+3ax^3+x^4\\
\hline
(a+x)^4=\underline{a^4+4a^3x+6a^2x^2+4ax^3+x^4}\text{ Ans.}
\end{array}$$

Again, in the preceding page, by writing a for x, and y for a, we have

$$\begin{array}{l}
(a-y)^3=a^3-3a^2y+3ay^2-y^3\\
(a-y)^2=a^2-2ay+y^2\\
\hline
\quad a^5-3a^4y+\ 3a^3y^2-\ a^2y^3\\
\qquad -2a^4y+\ 6a^3y^2-6a^2y^3+2ay^4\\
\qquad\qquad +\ \ a^3y^2-3a^2y^3+3ay^4-y^5\\
\hline
(a-y)^5=\underline{a^5-5a^4y+10a^3y^2-10a^2y^3+5ay^4-y^5}
\end{array}$$

RULE II.

Ex. 8. Although the rule prescribes the finding the coefficients separately, it must not be understood as absolutely necessary, being merely stated in those terms for the sake of perspicuity; generally the whole operation is represented in one line, thus

$$(a+x)^4=a^4+4a^3x+\frac{4.3}{2}a^2x^2+\frac{6.2}{3}ax^3+\frac{4.1}{4}x^4$$

Or performing the divisions and multiplications

$$(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4$$

And, in the same manner, we have

$$(a-x)^5=a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$$

Ex. 4. Here $(a+x)^6=$

$$a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6, \text{ and}$$

$$(a-y)^7=$$

$$a^7-7a^6y+21a^5y^2-35a^4y^3+35a^3y^4-21a^2y^5+7ay^6-y^7$$

Ex. 5. Here $(2+x)^5=$

$$2^5+5.2^4x+10.2^3x^2+10.2^2x^3+5.2x^4+x^5$$

Or, establishing the powers of 2, $=$

$$32+80x+80x^2+40x^3+10x^4+x^5$$

In the other part of this example the quantity is a trinomial, which it is better to put under a binomial form, thus:

$$\{(a-bx)+c\}^3=(a-bx)^3+3(a-bx)^2c+3(a-bx)c^2+c^3$$

Then, involving the several powers of $a-bx$, we have

$$(a-bx)^3 = a^3 -3a^2bx+3ab^2x^2-b^3x^3$$
$$3c\,(a-bx)^2=3a^2c-6acbx+3cb^2x^2$$
$$3c^2(a-bx) =3c^2a-3c^2bx$$
$$c^3=c^3$$

Whence by addition $(a-bx+c)^3=$

$$a^3+3a^2c+3c^2a+c^3-3a^2bx-6acbx-3c^2bx+3ab^2x^2$$
$$3cb^2x^2-b^3x^3 \text{ Ans.}$$

EVOLUTION.

CASE I.

To find the root of a simple quantity.

Ex. 3. Here $\sqrt{4a^2x^6}=\sqrt{4}\sqrt{a^2}\sqrt{x^6}=2ax^3$ Ans.

Ex. 4. Here $\sqrt[3]{-125a^3x^6}=\sqrt[3]{-125}\sqrt[3]{a^3}\sqrt[3]{x^6}=-5ax^2$

Ex. 5. Here $\sqrt[4]{256a^4x^8}=\sqrt[4]{256}\sqrt[4]{a^4}\sqrt[4]{x^8}=4ax^2$

Ex. 6. Here $\sqrt{\dfrac{4a^4}{9x^2y^2}}=\dfrac{\sqrt{4}\sqrt{a^4}}{\sqrt{9}\sqrt{x^2}\sqrt{y^2}}=\dfrac{2a}{3xy}$ Ans.

Ex. 7. Here $\sqrt[3]{\dfrac{8a^3}{125x^6}}=\dfrac{\sqrt[3]{8}\sqrt[3]{a^3}}{\sqrt[3]{125}\sqrt[3]{x^6}}=\dfrac{2a}{5x^2}$

Ex. 8. Here $\sqrt[5]{\dfrac{-32a^5x^{10}}{243}}=\dfrac{\sqrt[5]{-32}\sqrt[5]{a^5}\sqrt[5]{x^{10}}}{\sqrt[5]{243}}=\dfrac{-2ax^2}{3}$

CASE II.

To extract the square root of a compound quantity

Ex. 2. $a^4+4a^3x+6a^2x^2+4ax^3+x^4(a^2+2ax+x^2$ Ans.

$$\begin{array}{ll}
 & a^4 \\
\hline
2a^2+2ax) & 4a^3x+6a^2x^2 \\
 & 4a^3x+4a^2x^2 \\
\hline
2a^2+4ax+x^2) & 2a^2x^2+4ax^3+x^4 \\
 & 2a^2x^2+4ax^3+x^4 \\
\hline
\end{array}$$

Ex. 3. $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}(x^2-x+\frac{1}{4}$ Ans.

$$\begin{array}{ll}
 & x^4 \\
\hline
2x^2-x) & -2x^3+\frac{3}{2}x^2 \\
 & -2x^3+\ x^2 \\
\hline
2x^2-2x+\frac{1}{4}) & \frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{16} \\
 & \frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{16} \\
\hline
\end{array}$$

Ex. 4.

$$
\begin{array}{lll}
 & 4x^6-4x^4+12x^3+x^2-6x+9\,(2x^3-x+3 & \text{Ans} \\
 & 4x^6 & \\
\cline{2-2}
4x^3-x) & \quad -4x^4+12x^3+x^2 & \\
 & \quad -4x^4+\quad\ x^2 & \\
\cline{2-2}
4x^3-2x+3) & \qquad\quad 12x^3-6x+9 & \\
 & \qquad\quad 12x^3-6x+9 & \\
\cline{2-2}
\end{array}
$$

Ex. 5.

$$
\begin{array}{ll}
 & x^6+4x^5+10x^4+20x^3+25x^2+24x+16\,(x^3+2x^2 \\
 & x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad +3x+4 \\
\cline{2-2}
2x^3+2x^2) & \quad 4x^5+10x^4 \\
 & \quad 4x^5+\ \ 4x^4 \\
\cline{2-2}
2x^3+4x^2+3x) & \qquad 6x^4+20x^3+25x^2 \\
 & \qquad 6x^4+12x^3+\ \ 9x^2 \\
\cline{2-2}
2x^3+4x^2+6x+4) & \qquad\qquad 8x^3+16x^2+24x+16 \\
 & \qquad\qquad 8x^3+16x^2+24x+16 \\
\cline{2-2}
\end{array}
$$

Ex. 6.

$$
\begin{array}{ll}
 & a^2+b\,(a+\frac{b}{2a}-\frac{b^2}{8a^3}+\frac{b^3}{16a^5}\ \&c. \\
 & a^2 \\
\cline{2-2}
2a+\frac{b}{2a}) & b \\
 & b+\frac{b^2}{4a^2} \\
\cline{2-2}
2a+\frac{b}{a}-\frac{b^2}{8a^3}) & -\frac{b^2}{4a^2} \\
 & -\frac{b^2}{4a^2}-\frac{b^3}{8a^4}+\frac{b^4}{64a^6} \\
\cline{2-2}
 & \qquad\frac{b^3}{8a^4}-\frac{b^4}{64a^6} \\
\cline{2-2}
\end{array}
$$

Ex. 7. $1+1(1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-$ &c.

1

$2+\frac{1}{2})\quad 1$

$1+\frac{1}{4}$

$2+1-\frac{1}{8})\quad -\frac{1}{4}$

$-\frac{1}{4}-\frac{1}{8}+\frac{1}{64}$

$2+1-\frac{1}{4})\quad \frac{1}{8}-\frac{1}{64}$

CASE III.

To find any root of a compound quantity.

Ex. 3. $4a^2-12ax+9x^2(2a-3x$ Ans.

$4a^2$

$4a)-12ax$

$(2a-3x)^2=4a^2-12ax+9x^2$

Ex. 4. $a^2+2ab+2ac+b^2+2bc+c^2(a+b+c$

a^2

$2a)\ 2ab$

$(a+b)^2=a^2+2ab+b^2$

$2a)\quad 2ac=$difference

$(a+b+c)^2=a^2+2ab+2ac+b^2+2bc+c^2$

Therefore $a+b+c$ is the root sought.

Ex. 5. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1(x^2-2x+1$

x^6

$3(x^2)^2=3x^4)\quad -6x^5$

$(x^2-2x)^3=x^6-6x^5+12x^4-8x^3$

$3x^4)\quad 3x^4-12x^3=$difference

$(x^2-2x+1)^3=x^6-6x^5+15x^4-20x^3+15x^2-6x+1$

Ans. x^2-2x+1

Ex. 6. $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4(2a-3x$

$16a^4$

$4(2a)^3=32a^3) \quad -96a^3x$

$(2a-3x)^4=16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$

Ex. 7. $32x^5-80x^4+80x^3-40x^2+10x-1(2x-1$

$32x^5$

$5(2x)^4=80x^4) \quad -80x^4$

$(2x-1)^5=32x^5-80x^4+80x^3-40x^2+10x-1$

IRRATIONAL QUANTITIES, OR SURDS.

CASE I.

To reduce a rational quantity to the form of a surd.

Ex. 3. Here $(5)^2=25$, therefore $\sqrt{25}$ Ans.

Ex. 4. Here $(-3x)^3=-27x^3$, therefore $\sqrt[3]{-27x^3}$ Ans.

Ex. 5. Here $(-2a)^4=16x^4$, therefore $\sqrt[4]{16x^4}$ Ans.

Ex. 6. Here $(a^2)^5=a^{10}$, therefore $\sqrt[5]{a^{10}}$ Ans.

And $(\sqrt{a}+\sqrt{b})^2=a+b+2\sqrt{ab}$; therefore $\sqrt{(a+b+2\sqrt{ab})}$ the answer.

Again $\left(\dfrac{\sqrt{a}}{2a}\right)^2=\dfrac{a}{4a^2}=\dfrac{1}{4a}$; therefore $\sqrt{\dfrac{1}{4a}}$ Ans.

Also $\left(\dfrac{a}{b\sqrt{a}}\right)=\dfrac{a^3}{b^2a}=\dfrac{a}{b^2}$; therefore $\sqrt{\dfrac{a}{b^2}}$ Ans.

Note to the above Case.

Ex. 1. Here $5\sqrt{6}=\sqrt{25}\times\sqrt{6}=\sqrt{150}$ Ans.

Ex. 2. Here $\dfrac{1}{5}\sqrt{5a}=\sqrt{\dfrac{1}{25}}\times\sqrt{5a}=\sqrt{\dfrac{a}{5}}$ Ans.

Ex. 3. Here $\dfrac{2a}{3}\sqrt[3]{\dfrac{9}{4a^2}}=\sqrt[3]{\dfrac{8a^3}{27}}\times\sqrt[3]{\dfrac{9}{4a^2}}=\sqrt[3]{\dfrac{72a^3}{108a^2}}=$

$=\sqrt[3]{\dfrac{2a}{3}}$ Ans.

CASE II.

To reduce quantities of different indices to others that shall have a given index.

Ex. 2. Here $\frac{1}{2}\div\frac{1}{6}=\frac{1}{2}\times\frac{6}{1}=3$ 1st index.

$\frac{1}{3}\div\frac{1}{6}=\frac{1}{3}\times\frac{6}{1}=2$ 2nd index.

Hence $(5^3)^{\frac{1}{6}}$ and $(6^2)^{\frac{1}{6}}$, or $125^{\frac{1}{6}}$ and $36^{\frac{1}{6}}$ are the answers sought.

Ex. 3. Here $\frac{1}{2}\div\frac{1}{8}=\frac{1}{2}\times\frac{8}{1}=4$ 1st index.

$\frac{1}{4}\div\frac{1}{8}=\frac{1}{4}\times\frac{8}{1}=2$ 2nd index.

Therefore $(2^4)^{\frac{1}{8}}$ and $(4^2)^{\frac{1}{8}}=16^{\frac{1}{8}}$ and $16^{\frac{1}{8}}$.

Whence $16^{\frac{1}{8}}$ and $16^{\frac{1}{8}}$, the answer sought.

Ex. 4. Here $\frac{2}{1}\div\frac{1}{4}=\frac{2}{1}\times\frac{4}{1}=8$ 1st index.

$\frac{1}{2}\div\frac{1}{4}=\frac{1}{2}\times\frac{4}{1}=2$ 2nd index.

Therefore $(a^8)^{\frac{1}{4}}$ and $(a^2)^{\frac{1}{4}}$ are the quantities sought

Ex. 5. Here $\frac{1}{2}\div\frac{1}{8}=\frac{1}{2}\times\frac{8}{1}=4$ 1st index.

$1\div\frac{1}{8}=\frac{1}{1}\times\frac{8}{1}=8$ 2nd index.

Therefore $(a^4)^{\frac{1}{8}}$ and $(b^8)^{\frac{1}{8}}$ are the answers.

Note to the above Case.

Ex. 2. Here $\frac{1}{3}$ and $\frac{1}{4}=\frac{4}{12}$ and $\frac{3}{12}$

Hence $(4^4)^{\frac{1}{12}}$ and $(5^3)^{\frac{1}{12}}$ Ans.

Ex. 3. Here $\frac{1}{2}=\frac{3}{6}$ and $\frac{1}{3}=\frac{2}{6}$

Therefore $(a^3)^{\frac{1}{6}}$ and $(a^2)^{\frac{1}{6}}$ Ans.

Ex. 4. Here $\frac{1}{3}=\frac{4}{12}$ and $\frac{1}{4}=\frac{3}{12}$

Therefore $(a^4)^{\frac{1}{12}}$ and $(b^3)^{\frac{1}{12}}$ Ans.

Ex. 5. Here $\frac{1}{n}$ and $\frac{1}{m}$ reduce to $\frac{m}{mn}$ and $\frac{n}{mn}$

Therefore $(a^m)^{\frac{1}{mn}}$ and $(b^n)^{\frac{1}{mn}}$ Ans.

CASE III.

To reduce surds to their most simple forms.

Ex. 3. Here $\sqrt{125}=\sqrt{(25\times 5)}=5\sqrt{5}$ Ans.

Ex. 4. Here $\sqrt{294}=\sqrt{(49\times 6)}=7\sqrt{6}$ Ans.

Ex. 5. Here $\sqrt[3]{56}=\sqrt[3]{(8\times 7)}=2\sqrt[3]{7}$ Ans.

Ex. 6. Here $\sqrt[3]{192}=\sqrt[3]{(64\times 3)}=4\sqrt[3]{3}$ Ans.

Ex. 7. Here $7\sqrt{80}=7\sqrt{(16\times 5)}=28\sqrt{5}$ Ans.

Ex. 8. Here $9\sqrt[3]{81}=9\sqrt[3]{(27\times 3)}=27\sqrt[3]{3}$ Ans.

Ex. 9. Here, reducing the radical, we have

$$\sqrt{\frac{5}{6}}=\sqrt{\frac{5\times 6}{36}}=\frac{1}{6}\sqrt{(5\times 6)}=\frac{1}{6}\sqrt{30}$$

Therefore $\frac{3}{121}\sqrt{\frac{5}{6}}=\frac{3}{121}\times\frac{1}{6}\sqrt{30}=\frac{1}{242}\sqrt{30}$ Ans.

Ex. 10. Here by *Note* 2. $\frac{4}{7}\sqrt[3]{\frac{3}{16}}=\frac{4}{7}\sqrt[3]{\frac{3}{8\times 2}}=\frac{2}{7}\sqrt[3]{\frac{3}{2}}$

$$=\frac{2}{7\times 2}\sqrt[3]{(3\times 2^2)}=\frac{1}{7}\sqrt[3]{12}\text{ the Answer.}$$

Ex. 11. Here $\sqrt{98a^2x}=\sqrt{(49a^2\times 2x)}=7a\sqrt{2x}$ Ans.

Ex. 12. Here $\sqrt{(x^3-a^2x^2)}=\sqrt{\{x^2(x-a^2)\}}=x\sqrt{(x-a^2)}$

CASE IV.

To add surd quantities together.

Ex. 5. First $\sqrt{72}=\sqrt{(36\times 2)}=6\sqrt{2}$
And $\sqrt{128}=\sqrt{(64\times 2)}=8\sqrt{2}$

Ans. $14\sqrt{2}$

Ex. 6. Here $\sqrt{180}=\sqrt{(36\times 5)}=6\sqrt{5}$
Also $\sqrt{405}=\sqrt{(81\times 5)}=9\sqrt{5}$

Ans. $15\sqrt{5}$

Ex. 7. First $3\sqrt[3]{40}=3\sqrt[3]{(8\times 5)}=6\sqrt[3]{5}$
And $\sqrt[3]{135}=\sqrt[3]{(27\times 5)}=3\sqrt[3]{5}$

Ans. $9\sqrt[3]{5}$

Ex. 8. Here $4\sqrt[3]{54}=4\sqrt[3]{(27\times 2)}=12\sqrt[3]{2}$
And $5\sqrt[3]{128}=5\sqrt[3]{(64\times 2)}=20\sqrt[3]{2}$

Ans. $32\sqrt[3]{2}$

Ex. 9. Here $9\sqrt{243}=9\sqrt{(81\times 3)}=81\sqrt{3}$
And $10\sqrt{363}=10\sqrt{(121\times 3)}=110\sqrt{3}$

Ans. $191\sqrt{3}$

Ex. 10. $3\sqrt{\frac{2}{3}}=3\sqrt{\frac{6}{9}}= \;.\;.\;.\;.\;.\; \sqrt{6}$

$$7\sqrt{\frac{27}{50}}=\frac{21}{5}\sqrt{\frac{3}{2}}=\frac{21}{5}\sqrt{\frac{6}{4}}=\tfrac{21}{10}\sqrt{6}$$

The sum $=\tfrac{31}{10}\sqrt{6}$

Ex. 11. $12\sqrt[3]{\frac{1}{4}}=12\sqrt[3]{\frac{2}{8}}= \;.\;.\;.\;.\; 6\sqrt[3]{2}$

$$3\sqrt[3]{\frac{1}{32}}=\frac{3}{2}\sqrt[3]{\frac{1}{4}}=\frac{3}{2}\sqrt[3]{\frac{2}{8}}=\tfrac{3}{4}\sqrt[3]{2}$$

The sum $=6\tfrac{3}{4}\sqrt[3]{2}$

Ex. 12. $\frac{1}{2}\sqrt{a^2b} = \frac{1}{2}a\sqrt{b}$

$\frac{1}{3}\sqrt{4bx^4} = \frac{2}{3}x^2\sqrt{b}$

The sum $= \left(\frac{1}{2}a + \frac{2}{3}x^2\right)\sqrt{b}$

CASE V.

To find the difference of surd quantities.

Ex. 1. Here $2\sqrt{50} = 2\sqrt{(25\times 2)} = 10\sqrt{2}$
And $\sqrt{18} = \sqrt{(9\times 2)} = 3\sqrt{2}$

Difference $= 7\sqrt{2}$

Ex. 2. Here $\sqrt[3]{320} = \sqrt[3]{(64\times 5)} = 4\sqrt[3]{5}$
And $\sqrt[3]{40} = \sqrt[3]{(8\times 5)} = 2\sqrt[3]{5}$

Difference $= 2\sqrt[3]{5}$

Ex. 3. Here $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \frac{1}{5}\sqrt{15}$

And $\sqrt{\frac{5}{27}} = \sqrt{\frac{15}{81}} = \frac{1}{9}\sqrt{15}$

Difference $= \frac{4}{45}\sqrt{15}$

Ex. 4. Here $2\sqrt{\tfrac{1}{2}} = 2\sqrt{\tfrac{2}{4}} = \sqrt{2}$
And $\sqrt{8} = \sqrt{(4\times 2)} = 2\sqrt{2}$

Difference $= \sqrt{2}$

Ex. 5. Here $3\sqrt[3]{\frac{1}{3}} = 3\sqrt[3]{\frac{9}{27}} = \sqrt[3]{9}$

And $\sqrt[3]{72} = \sqrt[3]{(8\times 9)} = 2\sqrt[3]{9}$

Difference $= \sqrt[3]{9}$

Ex. 6. Here $\sqrt[3]{\frac{2}{3}}=\sqrt[3]{\frac{18}{27}}=\frac{1}{3}\sqrt[3]{18}$

And $\sqrt[3]{\frac{9}{32}}=\sqrt[3]{\frac{18}{64}}=\frac{1}{4}\sqrt[3]{18}$

Difference $=\frac{1}{12}\sqrt[3]{18}$

Ex. 7. Here $\sqrt{80a^4x}=\sqrt{(16a^4\times 5x)}=4a^2\sqrt{5x}$

And $\sqrt{20a^2x^3}=\sqrt{(4a^2x^2\times 5x)}=2ax\sqrt{5x}$

Difference $=(4a^2-2ax)\sqrt{5x}$

Ex. 8. Here $8\sqrt[3]{a^3b}=8\sqrt[3]{(a^3\times b)}=8a\sqrt[3]{b}$

And $2\sqrt[3]{a^6b}=2\sqrt[3]{(a^6\times b)}=2a^2\sqrt[3]{b}$

Difference $=(8a-2a^2)\sqrt[3]{b}$

CASE VI.

To multiply surd quantities together.

Ex. 5. Mult. $5\sqrt[3]{8}$

By $3\sqrt[3]{5}$

Product $=15\sqrt[3]{40}=15\sqrt[3]{(8\times 5)}=30\sqrt[3]{5}$ Ans.

Ex. 6. Mult. $\sqrt[3]{18}$

By $5\sqrt[3]{4}$

Prod. $=5\sqrt[3]{72}=5\sqrt[3]{(8\times 9)}=10\sqrt[3]{9}$

Ex. 7. Mult. $\frac{1}{4}\sqrt[3]{6}$

By $\frac{2}{15}\sqrt[3]{9}$

Prod. $=\frac{1}{30}\sqrt[3]{54}=\frac{1}{30}\sqrt[3]{(27\times 2)}=\frac{1}{10}\sqrt[3]{2}$

Ex. 8. Mult. $\frac{1}{2}\sqrt[3]{18}$

By $5\sqrt[3]{20}$

Prod. $=\frac{5}{2}\sqrt[3]{360}=\frac{5}{2}\sqrt[3]{(8\times 45)}=5\sqrt[3]{45}$

Ex. 9. Mult. $2\sqrt{3} = 2\sqrt[6]{27}$

By $13\frac{1}{2}\sqrt[3]{5} = 13\frac{1}{2}\sqrt[6]{25}$

$27\sqrt[6]{675}$ Answer.

Ex. 10. Here $72\frac{1}{4}a^{\frac{2}{3}} \times 120\frac{1}{2}a^{\frac{1}{4}} = \frac{289}{4} \times \frac{241}{2} \times a^{\frac{2}{3}} \times a^{\frac{1}{4}}$

$= \frac{69649}{8}a^{\frac{8}{12}} \times a^{\frac{3}{12}} = 8706\frac{1}{8}a^{\frac{11}{12}}$ Ans.

Ex. 11. Mult. $4 + 2\sqrt{2}$

By $2 - \sqrt{2}$

$8 + 4\sqrt{2}$

$-4\sqrt{2} - 4$

$8 - 4 = 4$ the Answer.

Ex. 12. Mult. $(a+b)^{\frac{1}{n}} = (a+b)^{\frac{m}{mn}}$

By $(a+b)^{\frac{1}{m}} = (a+b)^{\frac{n}{mn}}$

Product $= (a+b)^{\frac{m+n}{mn}}$

CASE VII.

To divide one surd quantity by another.

Ex. 5. Here $\frac{6\sqrt{54}}{3\sqrt{2}} = 2\sqrt{27} = 2\sqrt{(9 \times 3)} = 6\sqrt{3}$ Ans.

Ex. 6. Here $\frac{4\sqrt[3]{72}}{2\sqrt[3]{18}} = 2\sqrt[3]{4}$ Answer.

Ex. 7. Here $5\frac{3}{4} \div \frac{2}{3} = \frac{23}{4} \times \frac{3}{2} = \frac{69}{8}$

And $\sqrt{\frac{1}{135}} \div \sqrt{\frac{1}{5}} = \sqrt{\frac{1}{27}} = \sqrt{\frac{3}{81}} = \frac{1}{9}\sqrt{3}$

Therefore $\frac{69}{8} \times \frac{1}{9}\sqrt{3} = \frac{69}{72}\sqrt{3} = \frac{23}{24}\sqrt{3}$ is the quotient.

Ex. 8. First $3\frac{5}{7} \div 2\frac{2}{5} = \frac{26}{7} \times \frac{5}{12} = \frac{65}{42}$

And $\sqrt[3]{\frac{2}{3}} \div \sqrt[3]{\frac{3}{4}} = \sqrt[3]{(\frac{2}{3} \times \frac{4}{3})} = \sqrt[3]{\frac{8}{9}} = \sqrt[3]{\frac{8 \times 3}{27}} = \frac{2}{3}\sqrt[3]{3}$

Hence $\frac{65}{42} \times \frac{2}{3}\sqrt[3]{3} = \frac{65}{63}\sqrt[3]{3}$ the Answer.

Ex. 9. Here $4\frac{1}{2} \div 2\frac{2}{3} = \frac{9}{2} \times \frac{3}{8} = \frac{27}{16}$

And $\sqrt{a} \div \sqrt[3]{ab} = a^{\frac{1}{2}} \div a^{\frac{1}{3}}b^{\frac{1}{3}} = \frac{a^{\frac{1}{6}}}{b^{\frac{1}{3}}} = \frac{a^{\frac{1}{6}}}{b^{\frac{2}{6}}} = \left(\frac{a}{b^2}\right)^{\frac{1}{6}}$

Therefore $\frac{27}{16}\left(\frac{a}{b^2}\right)^{\frac{1}{6}}$ the Answer.

Ex. 10. Here $32\frac{2}{5} \div 13\frac{3}{4} = \frac{162}{5} \times \frac{4}{55} = \frac{648}{275}$

And $\sqrt{a} \div \sqrt[3]{a} = a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{3}{6} - \frac{2}{6}} = a^{\frac{1}{6}}$

Therefore $\frac{648}{275}a^{\frac{1}{6}}$ the Answer required.

Ex. 11. Here $9\frac{3}{8} \div 4\frac{9}{11} = \frac{75}{8} \times \frac{11}{53} = \frac{825}{424}$

And $a^{\frac{1}{n}} \div a^{\frac{1}{m}} = a^{\frac{m}{mn}} \div a^{\frac{n}{mn}} = a^{\frac{m-n}{mn}}$

Consequently $\frac{825}{424}a^{\frac{m-n}{mn}}$ the Answer.

Ex. 12. Here $\frac{\sqrt{20} + \sqrt{12}}{\sqrt{5} - \sqrt{3}} = \frac{2\sqrt{5} + 2\sqrt{3}}{\sqrt{5} - \sqrt{3}}$

$= 2\frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}} = 2\left(\frac{8 + 2\sqrt{15}}{2}\right)$

$= 8 + 2\sqrt{15}$ Answer.

Note to the above Case.

Ex. 3. Here $\frac{1}{a^2} = a^{-2}$ the Answer.

Ex. 4. Here $a^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}}$ the Answer.

Ex. 5. Here $\frac{1}{a+x}=\frac{1}{(a+x)^1}=(a+x)^{-1}$ Answer.

Ex. 6. Here $a(a^2-x^2)^{-\frac{1}{3}}=\frac{a}{(a^2-x^2)^{\frac{1}{3}}}$ Answer.

CASE VIII.

To involve or raise surd quantities to any power.

Ex. 3. Here $(3\sqrt[3]{3})^2=(3\times 3^{\frac{1}{3}})^2=9.3^{\frac{2}{3}}$ Answer.

Ex. 4. Here $(17\sqrt{21})^3=(17\times 21^{\frac{1}{2}})^3=17^3\times 21^{\frac{3}{2}}$

$$17^3\times 21\times 21^{\frac{1}{2}}=103173\times 21^{\frac{1}{2}}$$

Ex. 5. Here $(\frac{1}{6}\sqrt{6})^4=(\frac{1}{6}\times 6^{\frac{1}{2}})^4=\frac{1}{6^4}\times 6^2=\frac{1}{6^2}$

Therefore $\frac{1}{36}$ the Answer.

Ex. 6. Multiply $3+2\sqrt{5}$
By $3+2\sqrt{5}$

$$9+6\sqrt{5}$$
$$+6\sqrt{5}+4\sqrt{25}$$

Square $=9+12\sqrt{5}+20=29+12\sqrt{5}$

Ex. 7. Multiply $\sqrt{x}+3\sqrt{y}$
By $\sqrt{x}+3\sqrt{y}$

$$x+3\sqrt{xy}$$
$$+3\sqrt{xy}+9y$$

Square $=x+6\sqrt{xy}+9y$
Mult. by $\sqrt{x}+3\sqrt{y}$

$$x\sqrt{x}+6x\sqrt{y}+9y\sqrt{x}$$
$$+3x\sqrt{y}+18y\sqrt{x}+27y\sqrt{y}$$

Cube $=x\sqrt{x}+9x\sqrt{y}+27y\sqrt{x}+27y\sqrt{y}$

Ex. 8. Here Mult. $\sqrt{3}-\sqrt{2}$
By $\sqrt{3}-\sqrt{2}$

$$\begin{array}{r} 3-\sqrt{6} \quad\quad \\ -\sqrt{6}+2 \\ \hline \end{array}$$

$$\begin{array}{rl} \text{Square} = & 5-\ 2\sqrt{6} \\ & 5-\ 2\sqrt{6} \\ \hline & 25-10\sqrt{6} \\ & \quad -10\sqrt{6}+24 \\ \hline \text{4th power} = & 49-20\sqrt{6} \text{ the Answer.} \end{array}$$

CASE IX.

To find the roots of surd quantities.

Ex. 3 Here $\sqrt{10^3} = \sqrt{(10^2\times 10)} = 10\sqrt{10}$ Answer.

Ex. 4. Here $\sqrt[3]{\frac{8}{27}a^4} = \sqrt[3]{\frac{8}{27}(a^3\times a)} = \frac{2}{3}a\sqrt[3]{a}$ Answer.

Ex. 5. Here $\sqrt[4]{\frac{16}{81}a^{\frac{2}{3}}} = \frac{2}{3}\sqrt[4]{a^{\frac{2}{3}}} = \frac{2}{3}\sqrt{a^{\frac{1}{3}}}$ Answer.

Ex. 6. Here $\frac{a}{3}\sqrt{\frac{a}{3}} = (\frac{a}{3})^{1+\frac{1}{2}} = (\frac{a}{3})^{\frac{3}{2}}$

Consequently $\sqrt[3]{(\frac{a}{3}\sqrt{\frac{a}{3}})} = \sqrt[3]{(\frac{a}{3})^{\frac{3}{2}}} = (\frac{a}{3})^{\frac{1}{2}} =$

$(\frac{3a}{9})^{\frac{1}{2}} = \frac{1}{3}\sqrt{3a}$ the Answer.

Ex. 7. Here $x^2-4x\sqrt{a}+4a(x-2\sqrt{a}$ the Answer.

$$\begin{array}{rl} & x^2 \\ \hline 2x-2\sqrt{a}) & -4x\sqrt{a}+4a \\ & -4x\sqrt{a}+4a \end{array}$$

Ex. 8. Here $a+2\sqrt{ab}+b(\sqrt{a}+\sqrt{b}$ the Answer.

$$\begin{array}{rl} & a \\ \hline 2\sqrt{a}+\sqrt{b}) & 2\sqrt{ab}+b \\ & 2\sqrt{ab}+b \\ \hline \end{array}$$

CASE X.

To transform a binomial or residual surd into a general surd.

Ex. 4. First $(3-\sqrt{5})^2=9-6\sqrt{5}+5=14-6\sqrt{5}$ therefore $\sqrt{(14-6\sqrt{5})}$ the Answer.

Ex. 5. Here $(\sqrt{2}-2\sqrt{6})^2=2-4\sqrt{12}+24=26-8\sqrt{3}$ therefore $\sqrt{(26-8\sqrt{3})}$ Answer.

Ex. 6. Here $(4-\sqrt{7})^2=16-8\sqrt{7}+7=23-8\sqrt{7}$ Hence $\sqrt{(23-8\sqrt{7})}$ Answer.

Ex. 7. In examples involving cube-root radicals, it is useful to know the following form of the cube of a binomial: viz.

$$(a\pm b)^3=a^3\pm b^3+3ab(a\pm b)$$

Hence $(2\sqrt[3]{3}-\sqrt[3]{9})^3=24-9+6\sqrt[3]{27}(2\sqrt[3]{3}-\sqrt[3]{9})=$ $15+18(2\sqrt[3]{3}-\sqrt[3]{9})$

Consequently $\sqrt[3]{\{15+18(2\sqrt[3]{3}-\sqrt[3]{9})\}}$ is the general surd required.

CASE XI.

To extract the square root of a binomial surd

Ex. 3. Here $\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{3+\frac{1}{2}\sqrt{(36-20)}\}}$ $=\sqrt{(3+2)}$; and $\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{3-\frac{1}{2}\sqrt{36-20)}\}}$ $=\sqrt{(3-2)}$.

Hence $\sqrt{(3+2)}\pm\sqrt{(3-2)}=\sqrt{5}\pm1$ Answer.

Ex. 4. Here

$\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{\frac{23}{2}+\frac{1}{2}\sqrt{(529-448)}\}}=\sqrt{(\frac{23}{2}+\frac{9}{2})}$ and $\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{\frac{23}{2}-\frac{1}{2}\sqrt{(529-448)}\}}=$

$$\sqrt{(\frac{23}{2}-\frac{9}{2})}$$

Hence $\sqrt{(\frac{23}{2}+\frac{9}{2})}\pm\sqrt{(\frac{23}{2}-\frac{9}{2})}=4\pm\sqrt{7}$ Answer.

Ex. 5. Here
$\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{18+\frac{1}{2}\sqrt{(1296-1100)}\}}=$
$\sqrt{(18+7)}$; and $\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}=$
$\sqrt{(18-\frac{1}{2}\sqrt{(1296-1100)}\}}=\sqrt{(18-7)}$
Hence $\sqrt{(18+7)}\pm\sqrt{(18-7)}=5\pm\sqrt{11}$ Ans.

Ex. 6. Here
$\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{\frac{33}{2}+\frac{1}{2}\sqrt{(1089-864)}\}}$
$=\sqrt{(\frac{33}{2}+\frac{15}{2})}$; and $\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}$
$=\sqrt{\{\frac{33}{2}-\frac{1}{2}\sqrt{(1089-864)}\}}=\sqrt{(\frac{33}{2}-\frac{15}{2})}$

Hence $\sqrt{(\frac{33}{2}+\frac{15}{2})}\pm\sqrt{(\frac{33}{2}-\frac{15}{2})}=\sqrt{24}\pm3$ Ans.

CASE XII.

To find such a multiplier, or multipliers, as will make any binomial surd rational.

Ex. 5. Given surd $\sqrt{5}-\sqrt{x}$
Multiplier $\sqrt{5}+\sqrt{x}$
Product $5-x$ as required.

Ex. 6. Given surd $\sqrt{a}+\sqrt{b}$
Multiplier $\sqrt{a}-\sqrt{b}$
Product $a-b$ as required.

Ex. 7. Given surd $a+\sqrt{b}$
Multiplier $a-\sqrt{b}$
Product a^2-b the Answer.

Ex. 8. Given surd $1-\sqrt[3]{2a}$
Square of the terms $=1$ & $\sqrt[3]{4a^2}$
Product with sign changed $=+\sqrt[3]{2a}$
Therefore $1+\sqrt[3]{2a}+\sqrt[3]{4a^2}=$ multiplier.
Mult. by $1-\sqrt[3]{2a}$

$$1+\sqrt[3]{2a}+\sqrt[3]{4a^2}$$
$$-\sqrt[3]{2a}-\sqrt[3]{4a^2}-\sqrt[3]{8a^3}$$

Product $=1-\sqrt[3]{8a^3}=1-2a$ as required.

Ex. 9. Given surd $\sqrt[3]{3}-\frac{1}{2}\sqrt[3]{2}$
Square of the terms $\sqrt[3]{9}$ and $\frac{1}{4}\sqrt[3]{4}$
Product with sign changed $+\frac{1}{2}\sqrt[3]{6}$
Whence $\sqrt[3]{9}+\frac{1}{2}\sqrt[3]{6}+\frac{1}{4}\sqrt[3]{4}$ is the multiplier required.

CASE XIII.

To reduce a fraction whose denominator is either a simple or common surd, to another that shall have a rational denominator.

Ex. 4. Here $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}=\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}\times\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}}=\frac{\sqrt{42}-\sqrt{18}}{4}$ Ans.

Ex. 5. Here $\frac{x}{3+\sqrt{x}}=\frac{x}{3+\sqrt{x}}\times\frac{3-\sqrt{x}}{3-\sqrt{x}}=\frac{3x-x\sqrt{x}}{9-x}$ Ans.

Ex. 6. Here $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\times\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{a+b-2\sqrt{ab}}{a-b}$ Answer sought.

Ex. 7. Here, by the preceding rule, the multiplier for the denominator is $\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}$.

Whence $\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}=\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}\times\frac{\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}}{\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}}$
$=\frac{10(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25})}{7-5}=5(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25})=$ the Answer required.

Ex. 8. Here $\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}=\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}\times$
$\frac{\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100}}{\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100}}=\frac{\sqrt[3]{3}(\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100})}{9+10}$
$=\frac{\sqrt[3]{243}-\sqrt[3]{270}+\sqrt[3]{300}}{19}$ Ans.

Ex. 9. Here $\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}=\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}\times\frac{\sqrt[4]{4}-\sqrt[4]{5}}{\sqrt[4]{4}-\sqrt[4]{5}}=$
$\frac{4(\sqrt[4]{4}-\sqrt[4]{5})}{\sqrt{4}-\sqrt{5}}=\frac{4(\sqrt[4]{4}-\sqrt[4]{5})}{2-\sqrt{5}}\times\frac{2+\sqrt{5}}{2+\sqrt{5}}=\frac{4(\sqrt[4]{4}-\sqrt[4]{5})(2+\sqrt{5})}{4-5}$
$=4(\sqrt[4]{5}-\sqrt[4]{4})(2+\sqrt{5})$ Ans.

ARITHMETICAL PROPORTION AND PROGRESSION.

Ex. 3. Here the formula $s=(a+l)\times\frac{n}{2}$ becomes $s=$ $(1+1000)\times 500=500500$. Ans.

Ex. 4. Here the formula $s=\{2a+(n-1)d\}\frac{n}{2}$ becomes $\{2+(100\times 2)\}\times\frac{101}{2}=202\times\frac{101}{2}=101^2=10201$ the Ans.

Ex. 5. Here $s=(a+l)\frac{n}{2}=(1+24)\times 12=300$ Ans.

Ex. 6. Here the formula $l=a+(n-1)d$, becomes $l=$ $2+(365-1)2=2+728=730$ Ans.

Ex. 7. Here $s=\{2a-(n-1)d\}\frac{n}{2}=\{20-\left(20\times\frac{1}{3}\right)\}\times\frac{21}{2}$ $=(20-\frac{20}{3})\times\frac{21}{2}=\frac{40}{3}\times\frac{21}{2}=140$ Ans.

Ex. 8. Here the 1st term is $1+1=2$, and the last $100+100=200$, the number of terms 100.

Therefore $s=(a+l)\frac{n}{2}=(2+200)50=10100$ yards, or 5 miles 1300 yards, the Answer.

GEOMETRICAL PROPORTION AND PROGRESSION.

Ex. 3. Here the 1st term $a=1$, the ratio $r=2$, the number of terms $n=20$;

Whence the formula $s=\frac{a(r-1)}{r^n-1}$ becomes

$$\frac{1\times(2^{20}-1)}{2-1}=2^{20}-1=1048575 \text{ Ans.}$$

Ex. 4. Here $a=1$, $r=\frac{1}{2}$ and $n=8$, whence $s=$ $\frac{a(r^n-1)}{r-1}=\frac{1\times[1-(\frac{1}{2})^8]}{1-\frac{1}{2}}=\frac{2^8-1}{2^8\times\frac{1}{2}}=\frac{2^8-1}{2^7}=\frac{255}{128}=1\frac{127}{128}$ the sum required.

Ex. 5. Here $a=1$, $r=\frac{1}{3}$, and $n=10$; whence

$$s=\frac{a(r^n\sim 1)}{r\sim 1}=\frac{1\times[1-(\frac{1}{3})^{10}]}{1-\frac{1}{3}}=\frac{3^{10}-1}{3^{10}\times\frac{2}{3}}=\frac{3^{10}-1}{3^9\times 2}=$$

$$\frac{59049}{39366}=1\frac{6561}{13122} \text{ the Answer.}$$

Ex. 6. Here $a=1$, $r=2$, $n=32$. Whence

$$s=\frac{a(r^n\sim 1)}{r\sim 1}=\frac{1\times(2^{32}-1)}{2-1}=2^{32}-1=$$

4294967295 farthings = 4473924*l.* 5*s.* 3¾*d.*

EQUATIONS.

RESOLUTION OF SIMPLE EQUATIONS.

EXAMPLES FOR PRACTICE.

Ex. 1. Here $5x+22-2x=31$, by transposing
Gives $5x-2x$ or $3x=31-22=9$
Whence $x=\frac{9}{3}=3$ Ans.

Ex. 2. Here $4-9y=14-11y$, or
$11y-9y=14-4$, or $2y=10$,
Whence $y=\frac{10}{2}=5$ Ans.

Ex. 3. Here $x+18=3x-5$, or $3x-x=18+5$,
or $2x=23$, whence $x=11\frac{1}{2}$.

Ex. 4. Here $x+\frac{x}{2}+\frac{x}{3}=11$, or $\frac{6x}{6}+\frac{3x}{6}+\frac{2x}{6}=11$.

Mult. by 6. $6x+3x+2x=66$, or $11x=66$,
Whence $x=6$

Ex. 5. Multiply the given equation by 2, and we have $4x-x+2=10x-4$; whence $10x+x-4x=4+2$,

or $7x=6$, whence $x=\frac{6}{7}$

Ex 6. Mult. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{7}{10}$ by 60, gives

$30x+20x-15x=42$, or $35x=42$,

or $x=\frac{42}{35}=\frac{6}{5}=1\frac{1}{5}$ Ans.

Ex. 7. Mult. $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$, by 12, gives

$$6x+18+4x=48-3x+15, \text{ or}$$

$$13x=45, \text{ whence } x=\frac{45}{13}=3\frac{6}{13}$$

Ex. 8. Here $2+\sqrt{3x}=\sqrt{(4+5x)}$ being squared,
Gives $4+4\sqrt{3x}+3x=4+5x$
Whence $4\sqrt{3x}=5x-3x=2x$
Squaring, $48x=4x^2$, and dividing by $4x$,
we have $x=12$.

Ex. 9. Here $x+a=\frac{x^2}{x+a}$, or $x^2+2ax+a^2=x^2$

Whence $2ax=-a^2$, or $2x=-a$, or $x=-\frac{a}{2}$

Ex. 10. Here $\sqrt{x}+\sqrt{(a+x)}=\frac{2a}{\sqrt{(a+x)}}$

which multiplied by $\sqrt{(a+x)}$
Gives $\sqrt{(ax+x^2)}+a+x=2a$, or $\sqrt{(ax+x^2)}=a-x$,
Hence by squaring, $ax+x^2=a^2-2ax+x^2$

Conseq. $3ax=a^2$, or $3x=a$, or $x=\frac{a}{3}$

Ex. 11. Here $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$

Mult. by 12 gives $3ax-3b+4a=6bx-4bx+4a$
Whence $3ax-2bx=3b$, or $x(3a-2b)=3b$

Consequently $x=\frac{3b}{3a-2b}$ Ans.

Ex. 12. Here $\sqrt{(a^2+x^2)}=\sqrt[4]{(b^4+x^4)}$, by squaring,
Gives $a^2+x^2=\sqrt{(b^4+x^4)}$; squaring again
Gives $a^4+2a^2x^2+x^4=b^4+x^4$

Whence $2a^2x^2=b^4-a^4$, or $x=\sqrt{\frac{b^4-a^4}{2a^2}}$

Ex. 13. Here $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt{ax}$, by squaring,
Gives $2a+2\sqrt{(a^2-x^2)}=ax$,

Whence $\sqrt{(a^2-x^2)}=\frac{ax-2a}{2}$; squaring again,

$$a^2-x^2=\frac{a^2x^2-4a^2x+4a^2}{4}$$

Therefore $-4x^2=a^2x^2-4a^2x$, or dividing by x, and transposing $(a^2+4)x=4a^2$.

$\therefore x=\frac{4a^2}{a^2+4}$, the answer required.

Ex. 14. Here $\frac{a}{1+x}+\frac{a}{1-x}=b$, becomes by reduction

$$\frac{a-ax+a+ax}{1-x^2}=b, \text{ or } \frac{2a}{1-x^2}=b$$

Whence $2a=b-bx^2$, or $bx^2=b-2a$

Therefore $x=\sqrt{\frac{b-2a}{b}}$.

Ex. 15. Here $a+x=\sqrt{\{a^2+x\sqrt{(b^2+x^2)}\}}$
Squaring $a^2+2ax+x^2=a^2+x\sqrt{(b^2+x^2)}$
Therefore $2ax+x^2=x\sqrt{(b^2+x^2)}$
Divide by x, $2a+x=\sqrt{(b^2+x^2)}$
By squaring $4a^2+4ax+x^2=b^2+x^2$

Whence $4ax=b^2-4a^2$, or $x=\frac{b^2}{4a}-a$

the answer required.

Ex. 16. Multiplying the given equation by 2, we have $\sqrt{(x^2+3a^2)}+\sqrt{(x^2-3a^2)}=2x\sqrt{a}$.
By squaring $2x^2+2\sqrt{(x^4-9a^4)}=4ax^2$, or $\sqrt{(x^4-9a^4)}=2ax^2-x^2$
Squaring again $x^4-9a^4=4a^2x^4-4ax^4+x^4$, or

$$(4a-4a^2)x^4=9a^4, \text{ or } x=\sqrt[4]{\frac{9a^3}{4-4a}}$$

Ex. 17. Here $\sqrt{(a+x)}+\sqrt{(a-x)}=b$, by squaring, $2a+2\sqrt{(a^2-x^2)}=b^2$, or $\sqrt{(a^2-x^2)}=\frac{1}{2}b^2-a$,
Whence $a^2-x^2=\frac{1}{4}b^4-b^2a+a^2$

And $x=\sqrt{(b^2a-\frac{1}{4}b^4)}=\frac{b}{2}\sqrt{(4a-b^2)}$

Ex. 18. Given equation $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$

By cubing both sides after the form of Ex. 7, Case x, Surds, we have

$$2a+3\sqrt[3]{(a^2-x^2)}\left\{\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}\right\}=b^3.$$

But since $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$, this becomes

$$2a+3b\sqrt[3]{(a^2-x^2)}=b^3, \text{ or}$$

$$3\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{b}, \text{ or}$$

$$\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{3b}; \text{ whence}$$

$$a^2-x^2=\frac{(b^3-2a)^3}{27b^3}=\left(\frac{b^3-2a}{3b}\right)^3$$

$$\text{Therefore } x=\sqrt{\left\{a^2-\left(\frac{b^3-2a}{3b}\right)^3\right\}}$$

Ex. 19. Given the equation $\sqrt{a}+\sqrt{x}=\sqrt{ax}$

By transposition $\sqrt{ax}-\sqrt{x}=\sqrt{a}$

Or $\sqrt{x}(\sqrt{a}-1)=\sqrt{a}$,

Squaring $x(\sqrt{a}-1)^2=a$,

$$\text{And } \therefore x=\frac{a}{(\sqrt{a}-1)^2}$$

Ex. 20. Here $\sqrt{\frac{x+1}{x-1}}+\sqrt{\frac{x-1}{x+1}}=a$,

By adding the quantities on the left

$$\frac{x+1+x-1}{\sqrt{(x^2-1)}}=a,$$

$$\text{Or } \frac{2x}{\sqrt{(x^2-1)}}=a, \text{ or } 2x=a\sqrt{(x^2-1)}$$

Squaring $4x^2=a^2x^2-a^2$

Transposing $x^2(a^2-4)=a^2$,

$$\therefore x^2=\frac{a^2}{a^2-4}, \text{ and } x=\frac{a}{\sqrt{(a^2-4)}}$$

Ex. 21. This equation, when transposed, is

$\sqrt{(a^2+ax)}+\sqrt{(a^2-ax)}=a$, squaring

$2a^2+2\sqrt{(a^4-a^2x^2)}=a^2$; whence

$\sqrt{(a^4-a^2x^2)}=-\frac{1}{2}a^2$; squaring again,

$a^4-a^2x^2=\frac{1}{4}a^4$, or $a^2x^2=\frac{3}{4}a^4$,

Whence $x=\sqrt{\frac{3}{4}a^2}=\frac{a}{2}\sqrt{3}$

Ex. 22. Here the given equation

$$\sqrt{(a^2-x^2)}+x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}$$

becomes, by squaring,

$$a^2-x^2+2x\sqrt{(a^4-a^2x^2-a^2+x^2)}+a^2x^2-x^2=a^4-a^4$$

Whence, by transposition, we have

$$a^4-a^2x^2-a^2+x^2-2x\sqrt{(a^4-a^2x^2-a^2+x^2)}+x^2=a^4x^2$$

And by evolution

$$\sqrt{(a^4-a^2x^2-a^2+x^2)}-x=a^2x, \text{ or}$$

$$\sqrt{(a^4-a^2x^2-a^2+x^2)}=x(a^2+1)$$

Wherefore, by involution

$$a^4-a^2x^2-a^2+x^2=a^4x^2+2a^2x^2+x^2$$

Or, by transposition

$$a^4x^2+3a^2x^2=a^4-a^2$$

Whence we have

$$x^2=\frac{a^4-a^2}{a^4+3a^2}=\frac{a^2-1}{a^2+3}$$

Or $x=\sqrt{\left(\frac{a^2-1}{a^2+3}\right)}$, the Answer.

Ex. 23. Here $\sqrt{(x+a)}+\sqrt{(x+b)}=c$,

By transposition $\sqrt{(x+a)}=c-\sqrt{(x+b)}$,

Squaring $x+a=c^2-2c\sqrt{(x+b)}+x+b$

$\therefore 2c\sqrt{(x+b)}=c^2+b-a$

Or $\sqrt{(x+b)}=\frac{c^2+b-a}{2c}$

Squaring $x+b=\left(\frac{c^2+b-a}{2c}\right)^2$

And $\therefore x=\left(\frac{c^2+b-a}{2c}\right)^2-b$ the Answer.

Ex. 24. Here $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}$, by squaring,

$$\text{Gives } \frac{b}{a+x}+\frac{c}{a-x}+\sqrt{\frac{4bc}{a^2-x^2}}=\sqrt{\frac{4bc}{a^2-x^2}}$$

$$\text{Whence } \frac{b}{a+x}+\frac{c}{a-x}=0, \text{ or}$$

$$\frac{ab-bx+ac+cx}{a^2-x^2}=0,$$

$$\text{Therefore } ab-bx+ac+cx=0,$$

$$\text{And } bx-cx=ab+ac$$

$$\text{Consequently } x=\frac{ab+ac}{b-c}=a\left(\frac{b+c}{b-c}\right).$$

The Resolution of Simple Equations, containing two unknown Quantities.

EXAMPLES FOR PRACTICE.

Ex. 1. Here $4x+y=34$, and
$4y+x=16$.

Multiply the first equation by 4, and we have

$$4y+16x=136$$

$$\text{Subt. 2d } 4y+x=16$$

$$\text{Gives } 15x=120, \text{ or } x=\frac{120}{15}=8,$$

From 1st $y=34-4x=34-32=2$.

Whence 8 and 2 are the values required.

Ex. 2. Given $\left.\begin{matrix}2x+3y=16\\3x-2y=11\end{matrix}\right\}$

Mult. 1st by 3 gives $6x+9y=48$

2d by 2 gives $6x-4y=22$

Difference $\quad 13y=26$, or $y=2$

$$\text{From the 1st } x=\frac{16-3y}{2}=\frac{16-6}{2}=5.$$

Ex. 3. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$

1st Equat. mult. by 20, $8x+15y=9$

2d Equat. mult. by 20, $15x+8y=\frac{61}{6}=10\frac{1}{6}$

From the 1st of these $x=\frac{9-15y}{8}$

And from the 2d $x=\frac{10\frac{1}{6}-8y}{15}$

Hence $\frac{9-15y}{8}=\frac{10\frac{1}{6}-8y}{15}$ or

$135-225y=81\frac{1}{3}-64y$, or $161y=53\frac{2}{3}$,

Whence $y=53\frac{2}{3}\div 161=\frac{161}{3}\div 161=\frac{1}{3}$,

Also $x=\frac{9-15y}{8}=\frac{9-5}{8}=\frac{1}{2}$.

Ex. 4. Given $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$

Mult. 1st and 2d by 7, and we have

$$x+49y=693$$
$$49x+\quad y=357$$

Mult. the latter by 49, and we have

$$2401x+49y=17493$$

Subt. $\qquad x+49y=\quad 693$

Difference $2400x=16800$, whence $x=7$,

Also $y=357-49x=357-343=14$.

Ex. 5. Here $\frac{x}{2}-12=\frac{y}{4}+8$, and

$$\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27,$$

Mult. 1st by 4, and the 2d by 60

Then $2x-48=y+32$

$$12x+12y+20x-480=30y-15x+1620,$$

Whence $\left.\begin{array}{r}2x-\quad y=\quad 80\\ 47x-18y=2100\end{array}\right\}$

Mult. 1st by 18, gives $36x-18y=1440$

Sub. it from 2d $\qquad 47x-18y=2100$

Gives $11x=660$, or $x=60$

And $y=2x-80=120-80=40$.

Ex. 6. Given $x+y=s$, and $x^2-y^2=d$,
From the 1st $x=s-y$. 2d $x=\sqrt{(d+y^2)}$
Whence $s-y=\sqrt{(d+y^2)}$, or by squaring,

$$s^2-2sy+y^2=d+y^2,$$

Whence $2sy=s^2-d$, or $y=\frac{s^2-d}{2s}$

Also $x=s-y=s-\frac{s^2-d}{2s}=\frac{s^2+d}{2s}$

Otherwise, divide $x^2-y^2=d$
By $x+y=s$,
And we have $x-y=\frac{d}{s}$.

Whence by adding and subtracting the two latter, we obtain

$$2x=s+\frac{d}{s}, \text{ and } 2y=s-\frac{d}{s},$$

Whence $x=\frac{s^2+d}{2s}$ and $y=\frac{s^2-d}{2s}$, as before.

Ex. 7. Given $x+y:a::x-y:b$,
and $x^2-y^2=c$.

Multiplying the first and third terms of the proportion by $x-y$, we have

$$x^2-y^2:a::(x-y)^2:b,$$

$$\text{or } c:a::(x-y)^2:b,$$

$$\therefore x-y=\sqrt{\frac{bc}{a}}$$

$$\text{and } x+y=\frac{a}{b}(x-y)=\frac{a}{b}\sqrt{\frac{bc}{a}}.$$

$\therefore$ by addition and subtraction

$$2x=(\frac{a}{b}+1)\sqrt{\frac{bc}{a}}=\frac{a+b}{b}\sqrt{\frac{bc}{a}}=(a+b)\sqrt{\frac{c}{ab}}$$

$$2y=(\frac{a}{b}-1)\sqrt{\frac{bc}{a}}=\frac{a-b}{b}\sqrt{\frac{bc}{a}}=(a-b)\sqrt{\frac{c}{ab}}$$

$$\therefore x=\frac{a+b}{2}\sqrt{\frac{c}{ab}}, \text{ and } y=\frac{a-b}{2}\sqrt{\frac{c}{ab}}.$$

Ex. 8. Given $\left.\begin{array}{l} ax+by=c \\ dx+ey=f \end{array}\right\}$

Mult. 1st by d, $dax+dby=dc$
Mult. 2d by a, $dax+aey=af$
By subtrac. $dby-aey=dc-af$
Whence $y=\frac{dc-af}{db-ae}=\frac{af-dc}{ae-db}$
Mult. 1st by e, $eax+eby=ec$
Mult. 2d by b, $bdx+eby=bf$
By subtraction, $eax-bdx=ec-bf$
Whence $x=\frac{ec-bf}{ea-bd}=\frac{bf-ce}{bd-ea}$

Ex. 9. Given $x^2+y^2=a$, and $x^2-y^2=b$.

By addition $2x^2=a+b$, or $x^2=\frac{a+b}{2}$

$$\text{and} \therefore x=\sqrt{\frac{a+b}{2}}$$

By subtraction, $2y^2=a-b$, or $y^2=\frac{a-b}{2}$

$$\text{and} \therefore y=\sqrt{\frac{a-b}{2}}.$$

Ex. 10. Here $\left.\begin{array}{l} x^2+xy=a \\ y^2+xy=b \end{array}\right\}$

By add. $x^2+2xy+y^2=a+b$, or
$(x+y)^2=a+b$, or $x+y=\sqrt{(a+b)}$.

Now the two proposed equations may be put under the form

$$x(x+y)=a, \text{ or } x\{\sqrt{(a+b)}\}=a$$
$$y(x+y)=b, \text{ or } y\{\sqrt{(a+b)}\}=b$$

Whence, by division, $x=\frac{a}{\sqrt{(a+b)}}$; and

$$y=\frac{b}{\sqrt{(a+b)}}, \text{ the Answer.}$$

The Resolution of Simple Equations, containing three or more unknown Quantities.

PRACTICAL EXAMPLES

Ex. 3. Given
$$\begin{aligned} x+y+z&=53 \\ x+2y+3z&=105 \\ x+3y+4z&=134 \end{aligned}$$

Subtract 1st from 2d, $y+2z=52$
Subtract 2d from 3d, $y+\ z=29$

Now, subtracting the latter from the preceding one, we have $z=23$.

Also from the last $y=29-z=29-23=6$
And from the first $x=53-y-z=53-29=24$,
That is $x=24$, $y=6$, and $z=23$.

Ex. 4. Given
$$x+\tfrac{1}{2}y+\tfrac{1}{3}z=32$$
$$\tfrac{1}{3}x+\tfrac{1}{4}y+\tfrac{1}{5}z=15$$
$$\tfrac{1}{4}x+\tfrac{1}{5}y+\tfrac{1}{6}z=12$$

Multiplying the first by 6, and the second and third by 60, we have
$$6x+\ 3y+\ 2z=192$$
$$20x+15y+12z=900$$
$$15x+12y+10z=720$$

Again multiply the first by 10, the second by 3, and the third by 4, then
$$60x+30y+20z=1920$$
$$60x+45y+36z=2700$$
$$60x+48y+40z=2880$$

Subtracting the first of these from the 2d, and the second from the 3d, we have
$$15y+16z=780$$
$$3y+\ 4z=180$$

Mult. the latter by 5, $15y+20z=900$
Subtract . . . . $15y+16z=780$

$4z=120$, or $z=30$

But $y=\dfrac{180-4z}{3}=20$, and $x=\dfrac{192-3y-2z}{6}=12$

Therefore $x=12$, $y=20$, and $z=30$.

Ex. 5. Given
$$7x+5y+2z=\ 79$$
$$8x+7y+9z=122$$
$$x+4y+5z=\ 55$$

Multiply the first by 8, the second by 7, and the third by 56, and we have
$$56x+\ 40y+\ 16z=\ 632$$
$$56x+\ 49y+\ 63z=\ 854$$
$$56x+224y+280z=3080$$

Subtract the first from the 2d, and the second from the 3d, and we obtain

$$9y+47z=222$$
$$175y+217z=2226$$

Multiply the first of these by 175, and the second by 9, and we have

$$1575y+8225z=38850$$
$$1575y+1953z=20034$$

By subtraction $6272z=18816$, or $z=3$

But $y=\dfrac{222-47z}{9}=9$, and $x=55-4y-5z=4$

That is, $x=4$, $y=9$, $z=3$.

Ex. 6. Given $x+y=a$
$x+z=b$
$y+z=c$

By addition $2x+2y+2z=a+b+c$, or
$x+y+z=\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c$;

From which, subtracting each of the three given equations respectively, we have

$$z=-\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2}c$$
$$y=\tfrac{1}{2}a-\tfrac{1}{2}b+\tfrac{1}{2}c$$
$$x=\tfrac{1}{2}a+\tfrac{1}{2}b-\tfrac{1}{2}c$$

The values sought.

SIMPLE EQUATIONS.

Ex. 1. Let one of the parts $=x$,
Then the other will be $=15-x$.
And by the question $15-x=\frac{3x}{4}$, or
$60-4x=3x$, or $7x=60$; whence $x=\frac{60}{7}=8\frac{4}{7}$ the one part; and $15-x=6\frac{3}{7}$ the other.

Ex. 2. Let the value of the purse be x; then the money $=7x$; and by the question $7x+x=20$, or $8x=20$, or $x=\frac{20}{8}=2s.\ 6d.$, the value of the purse, and consequently $17s.\ 6d.$ the money contained in it.

Ex. 3. Let the number of sheep $=x$; then by the question, $x+x+\frac{1}{2}x+7\frac{1}{2}=500$; or $2\frac{1}{2}x=492\frac{1}{2}$; whence, multiplying by 2, we have $5x=985$; $\therefore x=\frac{985}{5}=197$, the number sought.

Ex. 4. Let the length of the post $=x$; then by the question $\frac{1}{4}x+\frac{1}{3}x+10=x$;
Whence, multiplying by 12, $3x+4x+120=12x$
Transposing, $5x=120$, or $x=\frac{120}{5}=24$ feet, the answer required.

Ex. 5. Let the number of guineas $=x$; then
$x-\frac{1}{4}x-\frac{1}{5}$ of $\frac{3}{4}x$, or $x-\frac{1}{4}x-\frac{3}{20}x=72$, or mult. by 20, $2\ 0-\ 5-3x=1440$; whence $12x=1440$, or $x=\frac{1440}{12}=120$ guineas.

Ex. 6. Let B's share $=x$, then by the question,
A's share $=2x$,
C's share $=3x$,
Consequently, $x+2x+3x=300$, or $6x=300$; whence $x=\frac{300}{6}=50l.$ B's share, $2x=100l.$ A's share, and $3x=150l.$ C's share.

Ex. 7. Let the age of the wife at the time of the marriage $=x$; and that of the husband $3x$;
Then after 15 years their ages will be $x+15$, and $3x+15$; and by the question $3x+15=2(x+15)$;
or $3x+15=2x+30$;
Whence, by transposing, $x=15$, the age of the wife; and $3x=45$, the age of the husband.

Ex. 8. Let the number sought $=x$; then by the question

$$\frac{2(x-5)}{3}=40;$$

Whence $2x-10=120$, or $2x=130$; and consequently $x=\frac{130}{2}=65$, the number sought.

Ex. 9. Let the less number of voters $=x$, then the greater number $=x+120$, and by the question

$$x+x+120=1296.$$

Whence $2x=1296-120=1176$; conseq. $x=588$ for one candidate, and $x+120=708$ for the other.

Ex. 10. Let x represent the age of C, then
$3x$ is the age of B, and
$6x$ the age of A.
But by the question, $x+3x+6x=10x=140$;
whence $x=\frac{140}{10}=14$ C's age, $3x=42$ B's age, and $6x=84=$ A's age.

Ex. 11. Let the equal sum laid out by each be x; then A leaves off with $x+126$, and B with $x-87$; and by the question.

$$x+126=2(x-87); \text{ or } x+126=2x-174,$$

Whence $x=300l.$ the first stock of each.

Ex. 12. Let the price of the harness $=x$; then the price of the horse $=2x$, and the price of the chaise $=6x$: and by the question $x+2x+6x=60l.$

$$\text{or } 9x=60;$$

Whence $x=\frac{60}{9}=6l.\ 13s.\ 4d.$ the value of the harness; $2x=13l.\ 6s.\ 8d.$ the value of the horse; and $6x=40l.$ the value of the chaise.

Ex. 13. Let x denote the number of beggars; then by the question $3x-8=$ the number of pence he had about him, which is also expressed by $2x+3$;

Whence $3x-8=2x+3$, or, by transposing, $x=11$, the number of beggars.

Ex. 14. Let x denote the value of the livery; then $x+8$ is the whole amount of his hire for the year, or for 12 months.

Hence, as $12 : 7 :: x+8 : \dfrac{7x+56}{12}$, the hire for 7 months. But, by the question, the servant received $x+2\frac{2}{3}$; whence $\dfrac{7x+56}{12}=x+2\frac{2}{3}$, or $7x+56=12x+32$;

And by transposition $5x=24$, or $x=\frac{24}{5}=4\frac{4}{5}l.=4l.\ 16s.$ the answer required.

In the preceding examples only one unknown quantity has been employed, but it will be more convenient, in some of the following questions, to use two or more unknown letters, according to the nature of the equation.

Ex. 15. Here let x denote the son's share, and y the daughter's.

Then the value of their shares will obviously have to each other the ratio as half-a-crown to a shilling; that is, as 5 to 2.

Hence, then, we have $x : y :: 5 : 2$

And $x+y=560$

From the first $2x=5y$, or $x=\frac{5y}{2}$

Whence from the second $\frac{5y}{2}+y=560$,

or $5y+2y=7y=1120$; $\therefore y=\frac{1120}{7}=160l.$

the daughter's share; and $x=\frac{5y}{2}=400l.$ the son's share.

Ex. 16. Here it may be observed, that every number consisting of two digits is equal to 10 times the digit in the tens place plus that in the units.

If therefore x be put for the former, and y for the latter, the number itself will be denoted by $10x+y$; and the number of the digits inverted by $10y+x$.

Hence by the question, $\begin{cases} 10x+y=4x+4y \\ 10x+y+18=10y+x \end{cases}$

From the first of these we have $3y=6x$, or $y=2x$;
and from the second $9x-9y=-18$, or $y=x+2$;

$$\therefore 2x=x+2, \text{ and } x=2,$$

Hence $y=2x=4$, and $\therefore 10x+y=24$ the number sought.

Ex. 17. Let x represent the equal income of each; then by the question A's yearly expenditure is $\frac{4x}{5}$, and B's $\frac{4x}{5}+50$;

Therefore in four years, B spends $\frac{16x}{5}+200$: which exceeds his income in the same time (viz. $4x$) by 100; hence we have the equation

$$\frac{16x}{5}+200=4x+100, \text{ or}$$

$$16x+1000=20x+500,$$

Whence $4x=500$, or $x=\frac{500}{4}=125l.$ the income of each.

Ex. 18. Let the number of persons in company be x, and the number of shillings each paid $=y$: then xy will be the whole reckoning.

Now had there been three persons more in company, viz. $(x+3)$ each would have paid $(y-1)$ shillings; whence we have

$$(x+3)(y-1)=xy;$$

And from the other condition of the question,

$$(x-2)(y+1)=xy;$$

Whence, from actual multiplication, these become

$$xy+3y-x-3=xy$$
$$xy-2y+x-2=xy$$

From the first, $x=3y-3$, and from the second, $x=2y+2$; $\therefore 3y-3=2y+2$, or $y=5$, the number of shillings each paid. And $x=3y-3=12$ the number of persons in company.

Ex. 19. Let x denote the money he had about him; then, by borrowing x, and spending one shilling, he had left $2x-1$.

Also, at the second tavern, after borrowing $2x-1$, he had $4x-2$; but spending one shilling out of it, he had left $4x-3$.

At the third tavern he borrowed $4x-3$, and then had $8x-6$; and after again spending one shilling, he had left $8x-7$.

At the fourth tavern, borrowing $8x-7$, he had $16x-14$; but after spending another shilling he had left $16x-15$; which by the question is equal to nothing.

Whence $16x-15=0$, or $x=\frac{15}{16}=0s.\ 11\frac{1}{4}d.$ the money he had at first.

Ex. 20. Let x and y denote the two parts; then by the question

$$x+\ y=75, \text{ and}$$
$$3x-7y=15$$

Mult. the first by 3,	$3x+3y=225$
Subtract the 2d	$3x-7y=\ 15$
And we shall have	$10y=210$

Whence $y=\dfrac{210}{10}=21$ the less part.

And $x=75-y=54$ the greater part.

Ex 21. Let $x=$ the whole quantity of the mixture, then $\frac{1}{2}x+25$ was the quantity of spirits, and $\frac{1}{3}x-5$ the quantity of water.

Which together make the whole x; therefore

$$\tfrac{1}{2}x+\tfrac{1}{3}x+20=x; \text{ or by mult. by } 6,$$

$$3x+2x+120=6x;$$

Whence, by transposing, $x=120$, and conseq.

$$\tfrac{1}{2}x+25=85 \text{ gallons of spirits,}$$

$$\tfrac{1}{3}x-\ \ 5=35 \text{ gallons of water.}$$

Ex. 22. Let $x=$ the number of guineas, and
$y=$ the number of moidores;
Then $21x=$ the shillings paid in guineas.
And $27y=$ the shillings paid in moidores.

Now the whole number of pieces used being 100, and the number of shillings paid being 2400, we have

$$x+\ \ y=\ 100$$

$$21x+27y=2400$$

Multiply the first by 27, and we shall have

$$27x+27y=2700$$

$$\text{But } 21x+27y=2400$$

Hence by subtraction, $6x=300$, or $x=50$; hence there were 50 guineas, and 50 moidores.

Ex. 23. Let x be the number of days; then $14x$ miles will be travelled by one, and $16x$ miles by the other.

Hence $30x=197$, or $x=\frac{197}{30}=6^{d}.\ 13^{h}\ \frac{3}{5}$, the time required.

Ex. 24. Let x denote the weight of the body; then $\frac{1}{2}x+9$ is the weight of the head; and by the question $x=\frac{1}{2}x+9+9$, or $\frac{1}{2}x=18$.

Whence $x=36$ the weight of the body, $\frac{1}{2}x+9=27$ the weight of the head, and 9 the weight of the tail; consequently $36+27+9=72$lbs. the weight of the fish.

Ex. 25. Let $\frac{x}{2}$, $\frac{x}{3}$, $\frac{x}{4}$, represent the three parts required; in which case the three latter conditions of the question will be answered;

For the first multiplied by 2, the second by 3, and the third by 4, will obviously be all equal to x, and therefore equal to each other.

Hence then it only remains to fulfil the equation

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=10,$$

Or multiplying by 12, to clear it of fractions,

$$6x+4x+3x=120,$$

Whence $13x=120$, or $x=\frac{120}{13}$

Therefore $\frac{x}{2}=\frac{60}{13}=4\frac{8}{13}$; $\frac{x}{3}=\frac{40}{13}=3\frac{1}{13}$, and $\frac{x}{4}=\frac{30}{13}=2\frac{4}{13}$ are the parts sought.

Ex. 26. Let $2x$, $3x$, and $4x$, be the three parts; then it is obvious that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, are equal to each other;

Wherefore there only remains the equation

$$2x+3x+4x=36,$$

Whence $9x=36$, or $x=\frac{36}{9}=4$,

Conseq. $2x=8$, $3x=12$, and $4x=16$, are the parts required.

Ex. 27. Let the value of the first horse be x, and of the second y; then by the question

$$x+50=2y$$
$$y+50=3x$$

Multiplying the former by 3, we have

$$3x+150=6y,$$

Which, combined with the latter, gives

$$y+50+150=6y;$$

Hence $5y=200$, and $\therefore y=40l.$, the value of the second horse; and $x=2y-50=30l.$, the value of the first.

Ex. 28. Let $x=$ A's money at first, and $y=$ B's; then, by the quest., $x+5=2(y-5)$

$$y+5=3(x-5)$$

By transposition these equations become

$$x-2y=-15 \quad (1)$$
$$3x-y=20 \quad (2)$$

Mult. (2) by 2, and $6x-2y=40$

Subtract (1) $\underline{x-2y=-15}$

$$\therefore 5x=55$$

Or $x=11s.$, A's money at first, and from (2) $y=3x-20=13s.$, B's money.

Ex. 29. Let x and y be the two numbers; then by the question

$$x-y : x+y :: 2 : 3$$
$$x+y : xy :: 3 : 5$$

Which, by multiplying extremes and means, give

$$3x-3y=2x+2y$$
$$5x+5y=\ \ 3xy$$

From the first $x=5y$; which substituted in the second, gives $\qquad 25y+5y=15y^2$

Hence, dividing by y, and transposing

$15y=30$; or $y=2$ one number,

And $x=5y=10$ the other.

Ex. 30. Let $x=$ the number of shillings he had at first; then by the question he lost $\frac{1}{4}x$, and therefore had $\frac{3x}{4}$ left; he then won back 3*s*. and now had

$\frac{3x}{4}+3$, or $\frac{3x+12}{4}$, of which he lost one-third, and had then two-thirds of it remaining, viz. $\frac{6x+24}{12}$; to which adding 2*s*. won, he had $\frac{6x+48}{12}$.

Then losing one-seventh of this, he had $\frac{6}{7}$ths of it, viz. $\frac{36x+288}{84}$ left; which by the question was 12*s*.;

Whence $\frac{36x+288}{84}=12$, or $36x+288=1008$;

And conseq. $36x=720$, or $x=20s$. the money he had at the beginning.

Ex. 31. Let x be the number of leaps the greyhound takes, and y the length of each; then by the question

$3:4::x:\frac{4x}{3}$, the number the hare takes, and

$3:2::y:\frac{2y}{3}$, the length of each;

But the hare being 50 of her leaps before the greyhound, he has to pass over $\frac{4x}{3}+50$ leaps of the hare.

Now then umber of the leaps of each, viz. x and $\frac{4x}{3}+50$,

multiplied by their respective lengths, will be obviously equal; viz. $xy=(\frac{4x}{3}+50)\frac{2y}{3}$

Or dividing each by y, and reducing

$$3x=\frac{8x}{3}+100, \text{ or } 9x-8x=300.$$

Whence $x=300$, the number of leaps the greyhound takes before he catches the hare.

Ex. 32. Let $x-2$ be the first part, $x+2$ the second; $\frac{x}{2}$ the third, and $2x$ the fourth; then the four last conditions will be satisfied, and it only remains to fulfil the equation

$$x-2+x+2+\frac{x}{2}+2x=90,$$

$$\text{Or } \frac{9x}{2}=90, \text{ and } \therefore x=20.$$

Hence $x-2=18$, $x+2=22$, $\frac{x}{2}=10$, and $2x=40$ are the four required parts.

Ex. 33. Let $x=$ his yearly income; then $\frac{x}{3}+10=$ what he spent, and $x-\frac{x}{3}-10=\frac{2x}{3}-10$ what he saved.

Whence $\frac{2x}{3}-10=\frac{x}{2}+15$, or $\frac{x}{6}=25$, and $\therefore x=150$ Ans.

Ex. 34. Let x be the number of days the man would be in drinking of it: then $\frac{1}{x}$ will be the quantity he drinks in a day, $\frac{1}{30}$ the quantity that the woman drinks in a day, and $\frac{1}{12}$ the quantity that they drink together in a day.

Whence the equation $\frac{1}{x}+\frac{1}{30}=\frac{1}{12}$,

Or $60+2x=5x$ by multg. by $60x$,

$\therefore 3x=60$, or $x=20$, the number of days sought.

Ex. 35. Let x be the number of men in the side of the less square, and $x+1$ the number in the side of the greater; then x^2 will be the whole number of men in the former, and $(x+1)^2=x^2+2x+1$ in the latter:

Whence x^2+34, and $x^2+2x+1-59$ will each express the whole number of men; from which we have this equation,

$$x^2+2x-58=x^2+34, \text{ or } 2x=58+34=92$$

Whence $x=\frac{92}{2}=46$; and consequently

$x^2+34=2150$, the whole number of men.

Ex. 36. Let x, y, and z be the number of days in which A, B, and C, respectively, would finish the work; then

A will do $\frac{1}{x}$ part of it in one day,

B will do $\frac{1}{y}$ part, and C $\frac{1}{z}$ part.

Whence, by the question, we shall have

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{8}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{9}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{10}$$

And consequently by addition

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{8}+\frac{1}{9}+\frac{1}{10}=\frac{121}{360}.$$

Or, by division,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{121}{720}.$$

From this, subtracting each of the three first equations, we have

$$\frac{1}{z}=\frac{31}{720}, \text{ or } z=\frac{720}{31}=23\frac{7}{31}=\text{days for C}$$

$$\frac{1}{y}=\frac{41}{720}, \text{ or } y=\frac{720}{41}=17\frac{23}{41}=\text{days for B}$$

$$\frac{1}{x}=\frac{49}{720}, \text{ or } x=\frac{720}{49}=14\frac{34}{49}=\text{days for A}$$

QUADRATIC EQUATIONS.

EXAMPLES FOR PRACTICE.

Ex. 1. Given $x^2-8x+10=19$; by transp.

$$x^2-8x=9$$

Whence $x=4\pm\sqrt{(16+9)}=4\pm5$,
Therefore $x=9$, or -1.

Ex. 2. Given $x^2-x-40=170$; by transp.

$$x^2-x=210$$

Whence $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+210)}=\frac{1}{2}\pm\sqrt{\frac{841}{4}}$,

$x=\frac{1}{2}\pm\frac{29}{2}=15$, or -14, the answer.

Ex. 3. Given $3x^2+2x-9=76$; by transp.

$3x^2+2x=85$, or $x^2+\frac{2}{3}x=\frac{85}{3}$; whence

$$x=-\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{85}{3})}=-\frac{1}{3}\pm\sqrt{\frac{256}{9}}$$

$=-\frac{1}{3}\pm\frac{16}{3}=5$, or $-\frac{17}{3}$, the answer.

Ex. 4. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$; by transposition,

$\frac{1}{2}x^2-\frac{1}{3}x=\frac{5}{8}$, or $x^2-\frac{2}{3}x=\frac{10}{8}$.

Whence $x=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{10}{8})}=\frac{1}{3}\pm\sqrt{\frac{49}{36}}$

$=\frac{1}{3}\pm\frac{7}{6}=1\frac{1}{2}$, or $-\frac{5}{6}$, the answer

Ex. 5. Given $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$

Mult. by 2, $x-\frac{2}{3}\sqrt{x}=44\frac{1}{3}$; then

$$\sqrt{x}=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{133}{3})}=\frac{1}{3}\pm\sqrt{(\frac{400}{9})}$$

Or $\sqrt{x}=\frac{1}{3}\pm\frac{20}{3}=7$, or $-\frac{19}{6}$,

Conseq. $x=7^2=49$, or $(-\frac{19}{6})^2=\frac{361}{36}$.

Ex. 6. Given $x+\sqrt{(5x+10)}=8$

By transposing, $\sqrt{(5x+10)}=8-x$,

By squaring, $5x+10=64-16x+x^2$,

Or $x^2-21x=-54$; therefore we have

$$x=\frac{21}{2}\pm\sqrt{(\frac{441}{4}-54)}=\frac{21}{2}\pm\sqrt{\frac{225}{4}},$$

That is, $x=\frac{21}{2}\pm\frac{15}{2}=18$, or 3, the answer.

Ex. 7. Given $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}=2$.

Here, since the first index is double the second, the equation is a quadratic; therefore by the rule

$$(10+x)^{\frac{1}{4}}=\frac{1}{2}\pm\sqrt{(\tfrac{1}{4}+2)}=\frac{1}{2}\pm\sqrt{\frac{9}{4}}=2 \text{ or } -1,$$

Whence $(10+x)^{\frac{1}{2}}=4$ or 1,

Or, $10+x=16$ or 1; and $x=6$ or -9.

Ex. 8. Given $2x^4-x^2+96=99$; by transp.

$$2x^4-x^2=3, \text{ or } x^4-\frac{1}{2}x^2=\frac{3}{2},$$

Whence $x^2=\frac{1}{4}\pm\sqrt{(\frac{1}{16}+\frac{3}{2})}=\frac{1}{4}\pm\sqrt{\frac{25}{16}}$,

Therefore $x^2=\frac{6}{4}$ or -1, and $x=\sqrt{\frac{6}{4}}=\frac{1}{2}\sqrt{6}$ or $\sqrt{-1}$.

Ex. 9. Given $x^6+20x^3-10=59$; by transp.

$x^6+20x^3=69$; whence

$x^3=-10\pm\sqrt{(100+69)}=-10\pm13=3$ or -23,

and $x=\sqrt[3]{3}$, or $\sqrt[3]{-23}$.

Ex. 10. Given $3x^{2n}-2x^n+3=11$; by transp.

$$3x^{2n}-2x^n=8, \text{ or } x^{2n}-\frac{2}{3}x^n=\frac{8}{3}$$

Whence $x^n=\frac{1}{3}+\sqrt{(\frac{1}{9}+\frac{8}{3})}=\frac{1}{3}\pm\sqrt{\frac{25}{9}}$,

Therefore $x^n=2$ or $-\frac{4}{3}$, and $x=2^{\frac{1}{n}}$ or $\left(-\frac{4}{3}\right)^{\frac{1}{n}}$.

Ex. 11. Given $5\sqrt[4]{x}-3\sqrt{x}=1\frac{1}{3}$, or

$3\sqrt{x}-5\sqrt[4]{x}=-1\frac{1}{3}$; then by division

We shall have $x^{\frac{1}{2}}-\frac{5}{3}x^{\frac{1}{4}}=-\frac{4}{9}$

Whence $x^{\frac{1}{4}}=\frac{5}{6}\pm\sqrt{(\frac{25}{36}-\frac{4}{9})}=\frac{5}{6}\pm\sqrt{\frac{9}{36}}$

$\therefore x^{\frac{1}{4}}=\frac{5}{6}\pm\frac{3}{6}=\frac{4}{3}$, or $\frac{1}{3}$,

Conseq. $x=(\frac{4}{3})^4=3\frac{13}{81}$, or $\frac{1}{81}$ Ans.

Ex. 12. Given $\frac{2}{3}x\sqrt{(3+2x^2)}=\frac{1}{2}+\frac{2}{3}x^2$

Mult. by $\frac{3}{2}$, and we have $x\sqrt{(3+2x^2)}=\frac{3}{4}+x^2$

And, by squaring, $3x^2+2x^4=\frac{9}{16}+\frac{3}{2}x^2+x^4$,

Whence $x^4+\frac{3}{2}x^2=\frac{9}{16}$; therefore by the rule

$x^2=-\frac{3}{4}\pm\sqrt{(\frac{9}{16}+\frac{9}{16})}=-\frac{3}{4}\pm\sqrt{\frac{18}{16}}$,

or $x^2=-\frac{3}{4}\pm\frac{3}{4}\sqrt{2}$, or $x=\frac{1}{2}\sqrt{(-3\pm3\sqrt{2})}$

Ex. 13. Given $x\sqrt{(\frac{6}{x}-x)}=\frac{1+x^2}{\sqrt{x}}$; mult. by $\sqrt{x}$,

$x\sqrt{(6-x^2)}=1+x^2$; or, by squaring each side,
$6x^2-x^4=1+2x^2+x^4$,

Whence $2x^4-4x^2=-1$, or $x^4-2x^2=-\frac{1}{2}$

Therefore $x^2=1\pm\sqrt{(1-\frac{1}{2})}=1\pm\frac{1}{2}\sqrt{2}$,

Conseq. $x=\sqrt{(1\pm\frac{1}{2}\sqrt{2})}$ Ans.

Ex. 14. Given $\frac{1}{x}\sqrt{(1-x^3)}=x^2$; multiplying by x, we have $\sqrt{(1-x^3)}=x^3$, or $1-x^3=x^6$ or $x^6+x^3=1$; whence also we have,

$$x^3=-\frac{1}{2}\pm\sqrt{(\frac{1}{4}+1)}=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}\text{; and}$$

Consequently $x=(-\frac{1}{2}\pm\frac{1}{2}\sqrt{5})^{\frac{1}{3}}$

Ex. 15. Given $x\sqrt{(\frac{a}{x}-1)}=\sqrt{(x^2-b^2)}$. By squaring

$$ax-x^2=x^2-b^2\text{, or } 2x^2-ax=b^2$$

That is, $x^2-\frac{a}{2}x=\frac{b^2}{2}$, or $x=\frac{a}{4}\pm\frac{1}{4}\sqrt{(a^2+8b^2)}$.

Ex. 16. Given $\sqrt{(1+x-x^2)}-2(1+x-x^2)=\frac{1}{9}$;

Here $(1+x-x^2)-\frac{1}{2}(1+x-x^2)^{\frac{1}{2}}=-\frac{1}{18}$,

Therefore $(1+x-x^2)^{\frac{1}{2}}=\frac{1}{4}\pm\sqrt{(\frac{1}{16}-\frac{1}{18})}$

$$=\frac{1}{4}\pm\sqrt{\frac{2}{16\,.\,18}}=\frac{1}{4}\pm\sqrt{\frac{1}{144}}=\frac{1}{4}\pm\frac{1}{12}=\frac{1}{3}\text{, or }\frac{1}{6},$$

Consequently $1+x-x^2=(\frac{1}{3})^2=\frac{1}{9}$, or $(\frac{1}{6})^2=\frac{1}{36}$

From the 1st we have $x^2-x=\frac{8}{9}$, which gives

$$x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+\frac{8}{9})}=\frac{1}{2}\pm\frac{1}{6}\sqrt{41}$$

From the 2d we have $x^2-x=\frac{35}{36}$, which gives

$$x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+\frac{35}{36})}=\frac{1}{2}\pm\frac{1}{3}\sqrt{11}$$

Ex. 17. Given $\sqrt{(x-\frac{1}{x})}+\sqrt{(1-)\frac{1}{x}}=x$,

By trans. $\sqrt{(x-\frac{1}{x})}=x-\sqrt{(1-\frac{1}{x})}$,

Square both sides, and

$$x-\frac{1}{x}=x^2-2\sqrt{(x^2-x)}+1-\frac{1}{x},$$

$$\text{or } (x^2-x)-2\sqrt{(x^2-x)}+1=0,$$

Extract the square root, and

$$\sqrt{(x^2-x)}-1=0$$

$$\therefore x^2-x=1$$

$$\therefore x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+1)}=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}.$$

Ex. 18. Given $x^{4n}-2x^{3n}+x^n=6$.

This equation is the same as

$$(x^{4n}-2x^{3n}+x^{2n})-x^{2n}+x^n=6,$$

$$\text{or } (x^{2n}-x^n)^2-(x^{2n}-x^n)=6,$$

which is a quadratic; wherefore

$$x^{2n}-x^n=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+6)}=\frac{1}{2}\pm\frac{5}{2}=3 \text{ or } -2$$

Taking the equation $x^{2n}-x^n=3$, we have

$$x^n=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+3)}=\frac{1}{2}\pm\sqrt{\frac{13}{4}}=\frac{1}{2}\pm\frac{1}{2}\sqrt{13},$$

$$\text{and } \therefore x=\sqrt[n]{(\frac{1}{2}\pm\frac{1}{2}\sqrt{13})}.$$

The equation $x^{2n}-x^n=-2$, gives

$$x^n=\frac{1}{2}\pm\sqrt{(\frac{1}{4}-2)}=\frac{1}{2}\pm\sqrt{-\frac{7}{4}}=\frac{1}{2}\pm\frac{1}{2}\sqrt{-7}$$

$$\text{and } \therefore x=\sqrt[n]{(\frac{1}{2}\pm\frac{1}{2}\sqrt{-7})}.$$

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

QUESTIONS FOR PRACTICE.

Ex. 1. Let x and y represent the two parts; then we have $x+y=40$ and $x^2+y^2=818$,

From the 1st $x=40-y$, or $x^2=1600-80y+y^2$

By substit. $1600-80y+2y^2=818$, or

$2y^2-80y=-782$, or $y^2-40y=-391$;

Whence $y=20+\sqrt{(400-391)}=20+3=23$,
And $x=40-y=17$, the answer.

Ex. 2. Let x represent the number sought;
Then by the question $(10-x)x=21$, or $x^2-10x=-21$;
Whence $x=5\pm\sqrt{(25-21)}=5\pm2=7$ or 3.

Ex. 3. Let $12+x$ and $12-x$ be the two parts, their sum being 24, and difference $2x$; hence, per quest.,

$$(12+x)\times(12-x)=35\times2x,$$
$$\text{or } 144-x^2=70x,$$
$$\therefore x^2+70x=144,$$

And $x=-35+\sqrt{(35^2+144)}=-35+37=2$.

Consequently, $12+x=14$, and $12-x=10$, are the parts required.

Ex. 4. Let $x=$ less part, and $\therefore 20-x=$ greater; then, by the quest.,

$2(20-x)^2-3x^2=96$, which reduces to $x^2+80x=704$.

Hence $x=-40+\sqrt{(1600+704)}=-40+48=8$, the less part, and $20-x=12$, the greater part.

Ex. 5. Let $30-x$ and $30+x$ be the parts, making together 60, the number to be divided; hence, per question,

$$(30-x)\times(30+x) : (30-x)^2+(30+x)^2 :: 2 : 5$$
$$\text{or } 900-x^2 : 1800+2x^2 :: 2 : 5;$$

hence, multiplying means and extremes,

$$4500-5x^2=3600+4x^2;$$

$\therefore$ by trans. $9x^2=900$, or $x^2=100$,
and $\therefore x=10$, so that the two parts are $30-x=20$, and $30+x=40$.

Ex. 6. Here, in order to avoid radicals, let us assume x^2 and y^2 for the two parts; then, by the question,

$$x^2+y^2=146, \text{ and } x-y=6.$$

Which may now be solved the same as Ex. 1. Another method is as follows:

By squaring the second,	$x^2-2xy+y^2=36$
Subt. it from twice the first,	$2x^2+2y^2=292$
And we have	$x^2+2xy+y^2=256$
Hence by extracting	$x+y=16$
But	$x-y=6$

Whence, by addition, $2x=22$, or $x=11$ and $x^2=121$,
And by subtraction $2y=10$, or $y=5$ and $y^2=25$.

Ex. 7. Let x and y represent the two numbers,
Then by the question $x+y=23$
And $xy=116\frac{1}{4}$
Squaring the 1st, $x^2+2xy+y^2=529$
Subtracting 4 times the 2d, $4xy \quad =465$
And we have $x^2-2xy+y^2=\ 64$
Whence $x-y=8$; and since also $x+y=23$
We have, by addition, $2x=31$, or $x=15\frac{1}{2}$
And by subtraction $2y=15$, or $y=\ 7\frac{1}{2}$

Ex. 8. Let x and y be the two numbers, and consequentl $\frac{1}{x}$ and $\frac{1}{y}$ their reciprocals. Then by the quest. $x+y=\frac{4}{3}$ and $\frac{1}{x}+\frac{1}{y}=\frac{16}{5}$; which latter becomes $x+y=\frac{16xy}{5}$; whence $\frac{16xy}{5}=\frac{4}{3}$, or $xy=\frac{20}{48}=\frac{5}{12}$

Consequently $x=\frac{5}{12y}$; which, substituted in the firs gives $\frac{5}{12y}+y=\frac{4}{3}$, or $5+12y^2=16y$, or $y^2-\frac{4}{3}y=-\frac{5}{12}$

Whence $y=\frac{2}{3}+\sqrt{(\frac{4}{9}-\frac{5}{12})}=\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$,

and $x=\frac{4}{3}-y=\frac{1}{2}$

Ex. 9. Let x and y represent the two numbers,

Then by the question $x-y=15$, and $\frac{xy}{2}=y^3$

The second equat. gives $xy=2y^3$, or $x=2y^2$,
Whence by substitution in the first we have

$$2y^2-y=15, \text{ or } y^2-\frac{1}{2}y=\frac{15}{2}; \text{ and hence}$$

$$y=\frac{1}{4}+\sqrt{(\frac{1}{16}+\frac{15}{2})}=\frac{1}{4}+\frac{11}{4}=3,$$

Consequently, $x=15+y=15+3=18$.

Ex. 10. Let x and y be the two numbers; then by the question $x-y=5$ and $x^3-y^3=1685$.

By the 1st $x=5+y$, or $x^3=125+75y+15y^2+y^3$,
Consequently $125+75y+15y^2+y^3-y^3=1685$,
That is, by dividing by 15, $y^2+5y=104$;

Whence $y=-\frac{5}{2}+\sqrt{(\frac{25}{4}+104)}$, or $y=-\frac{5}{2}+\frac{21}{2}$

$=8$; therefore $x=5+y=5+8=13$.

Consequently 8 and 13 are the numbers required.

Ex. 11. Let x be the number of pieces, and y the shillings that each piece cost; then by the question

$$xy=675, \text{ and } 48x=675+y.$$

From the 1st, $y=\frac{675}{x}$; whence by substitution we have

$$48x=675+\frac{675}{x}, \text{ or } 48x^2-675x=675,$$

$$\text{Or } x^2-\frac{225}{16}x=\frac{225}{16}; \text{ whence}$$

$$x=\frac{225}{32}+\sqrt{(\frac{225^2}{32^2}+\frac{225}{16})}=\frac{225}{32}+\frac{255}{32}=15, \text{ the}$$

number of pieces required.

Ex. 12. Let x and y represent the two numbers; then by the question $x(x+y)=77$, or $x^2+xy=77$,

And $y(x-y)=12$, or $xy-y^2=12$.

By subtraction we have $x^2+y^2=65$; and the second equation gives $x=\frac{12+y^2}{y}$, or $x^2=\frac{144+24y^2+y^4}{y^2}$: hence the equation

$$\frac{144+24y^2+y^4}{y^2}+y^2=65$$

$$\therefore 144+24y^2+2y^4=65y^2$$

$$\text{or } y^4-\frac{41}{2}y^2=-72$$

$$\therefore y^2=\frac{41}{4}+\sqrt{\left(\frac{41}{4}\right)^2-72}=\frac{41}{4}+\frac{23}{4}=16$$

Hence $y=4$, and $x=\sqrt{(65-y^2)}=7$.

Ex. 13. Let x represent the number of sheep, and y the shillings each cost, and consequently $y+2$ what each sold for; then by the question we have

$$\left.\begin{aligned} xy=1200 \\ \text{And } (x-15)(y+2)=1080 \end{aligned}\right\};$$

The latter gives $xy+2x-15y=1110$:

Or, since $xy=1200$, we have $2x-15y=-90$.

But $y=\dfrac{1200}{x}$, and $15y=\dfrac{18,000}{x}$; which, subst[d] in the last equation, gives $2x-\dfrac{18,000}{x}=-90$, or $2x^2-18,000=-90x$, or $x^2+45x=9000$,

Whence $x=-\dfrac{45}{2}+\sqrt{(\dfrac{45^2}{4}+9000)}=-\dfrac{45}{2}+\dfrac{195}{2}=75$, the number of sheep.

Ex. 14. Let x and y be the two numbers; then by the question

$$\left.\begin{aligned} xy=x^2-y^2 \\ x^2+y^2=x^3-y^3 \end{aligned}\right\}$$

Make now $x=yz$, and these equations become

$$\left.\begin{aligned} y^2z=y^2z^2-y^2 \\ y^2z^2+y^2=y^3z^3-y^3 \end{aligned}\right\}$$

Divide both by y^2, and we shall have

$$\left.\begin{aligned} z=z^2-1 \\ z^2+1=(z^3-1)y \end{aligned}\right\}$$

From the first of the two latter we have

$$z^2-z=1, \text{ or } z=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5},$$

Consequently $z^2+1=z+2=\frac{5}{2}+\frac{1}{2}\sqrt{5}$,

And $(z^3-1)=(\frac{1}{2}+\frac{1}{2}\sqrt{5})^3-1=1+\sqrt{5}$,

Whence from the second of the two latter, viz. $z^2+1=(z^3-1)y$, we have $y=\dfrac{z^2+1}{z^3-1}=\dfrac{\frac{5}{2}+\frac{1}{2}\sqrt{5}}{1+\sqrt{5}}=\frac{1}{2}\sqrt{5}$,

And $x=zy=(\frac{1}{2}+\frac{1}{2}\sqrt{5})\times\frac{1}{2}\sqrt{5}=\frac{1}{4}(\sqrt{5}+5)$;

That is, $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(\sqrt{5}+5)$ are the numbers sought.

Ex. 15. Let $(x+z)$ and $(x-z)$ represent the two umbers,

Then by the question $(x+z)-(x-z)=2z=8$

And $(x+z)^4-(x-z)^4=14,560$

Now $(x+z)^4=x^4+4x^3z+6x^2z^2+4xz^3+z^4$

$(x-z)^4=x^4-4x^3z+6x^2z^2-4xz^3+z^4$

Whence by subtraction $8x^3z+8xz^3=14,560$,

Or by division, and substituting $z=4$, we have

$$x^3+16x=455$$

Mult. by x, $x^4+16x^2=455x=65\times 7x$.

Add $49x^2$ to both sides, and it becomes

$$x^4+65x^2=49x^2+65\times 7x$$

Therefore, by completing the square,

$$x^4+65x^2+\frac{65^2}{4}=(7x)^2+65(7x)+\frac{65^2}{4}$$

Whence $x^2+\dfrac{65}{2}=7x+\dfrac{65}{2}$, or $x^2=7x$, or $x=7$;

Conseq. $x+z=7+4=11$ one number,

And $x-z=7-4=3$ the other.

Ex. 16. Let x be the whole number of persons, and y the number of shillings each would have had to pay; then, after two were gone, the number was only $(x-2)$, and each person's share $y+10$. Now by the question

$$xy=175, \text{ and } (x-2)(y+10)=175,$$

From the latter $xy+10x-2y-20=175$, or

Since $xy=175$, we have $10x-2y=20$, or $5x-y=10$

But $y=\dfrac{175}{x}$; therefore $5x-\dfrac{175}{x}=10$, or

$$5x^2-10x=175, \text{ or } x^2-2x=35,$$

Whence $x=1+\sqrt{(1+35)}=7$, the number sought

Ex. 17. Let x be the number of persons at first, and y the shillings each would have received: then $x+2$ was the number at last, and $y-1$ what each actually received: hence the following equations

$$xy=144, \text{ and } (x+2)(y-1)=144,$$

From the latter $xy+2y-x-2=144$, or since $xy=144$

$2y-x=2$; from the first $y=\dfrac{144}{x}$; therefore $\dfrac{288}{x}-x=2$;

$$\text{or } x^2+2x=288$$

Conseq. $x=-1+\sqrt{(288+1)}=-1+17=16$ Answer

Ex. 18. Let $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$, represent any four numbers in geometrical progression; then we have

$$\frac{x^2}{y}+x+y+\frac{y^2}{x}=15=a$$

$$\frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=85=b$$

Make $x+y=s$, and $xy=r$; then will

$$x^2+y^2=s^2-2r,$$

$$\frac{x^2}{y}+\frac{y^2}{x}=a-s,$$

$$\frac{x^4}{y^2}+\frac{y^4}{x^2}=(a-s)^2-2r.$$

Here the first and third equations are derived from this consideration, that the sum of the squares of any two quantities is equal to the square of their sum minus twice their product. Whence, by adding the first and third, we have

$$\frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=s^2+(a-s)^2-4r$$

$$\therefore s^2+(a-s)^2-4r=b \quad . \quad . \quad . \quad (a)$$

And from the second we have $x^3+y^3=xy(a-s)$

or $x^3+y^3=r(a-s)$.

Also, since $x^3+y^3=(x+y)^3-3xy(x+y)$ or $=s^3-3rs$,

We have $r(a-s)=s^3-3rs$, or $r=\frac{s^3}{2s+a}$,

Which value, substituted for r in (a), gives

$$s^2+(a-s)^2-\frac{4s^3}{2s+a}=b, \text{ or}$$

$$2s^2-2as+a^2-\frac{4s^3}{2s+a}=b,$$

Now, by reduction and transposition,

$$2as^2+2bs=a^3-ab, \text{ or}$$

$$s^2+\frac{b}{a}s=\frac{1}{2}a^2-\frac{1}{2}b,$$

Consequently $s=-\frac{b}{2a}+\sqrt{(\frac{b^2}{4a^2}+\frac{1}{2}a^2-\frac{1}{2}b)}$,

Where, by substituting $a=15$ and $b=85$, we obtain

$$s=6, \text{ and } r=\frac{s^3}{2s+a}=\frac{216}{27}=8.$$

Hence then $x+y=6$, and $xy=8$; from which are determined $x=2$, and $y=4$; and therefore the numbers sought will be 1, 2, 4, 8.

Ex. 19. Let $a+z$ be one of the numbers, and $a-z$ the other, then we shall have

$$(a+z)+(a-z)=11, \text{ or } 2a=11$$
$$(a+z)^5+(a-z)^5=17831=b.$$

Now $(a+z)^5=a^5+5a^4z+10a^3z^2+10a^2z^3+5az^4+z^5$
and $(a-z)^5=a^5-5a^4z+10a^3z^2-10a^2z^3+5az^4-z^5$.

Consequently by addition we have

$$10az^4+20a^3z^2+2a^5=b, \text{ or}$$
$$z^4+2a^2z^2=\frac{b-2a^5}{10a}$$

Or, substituting the values of a and b, we have

$$z^4+\frac{121}{2}z^2=\frac{24849}{176},$$

$$\text{Whence } z^2=-\frac{121}{4}+\sqrt{(\frac{14641}{16}+\frac{24849}{176})}=-\frac{121}{4}+\frac{65}{2}=\frac{9}{4}$$

$$\text{or } z=\frac{3}{2}; \text{ consequently}$$

$$a+z=5\tfrac{1}{2}+1\tfrac{1}{2}=7, \text{ and } a-z=5\tfrac{1}{2}-1\tfrac{1}{2}=4,$$

That is, the two numbers are 4 and 7.

Ex. 20. Let $x-3y$, $x-y$, $x+y$, and $x+3y$, be the four numbers; then by the question

$$(x-3y)(x-y)(x+y)(x+3y)=176985=a,$$

or $(x^2-9y^2)(x^2-y^2)=a$, or $x^4-10x^2y^2+9y^4=a$;

Or, since by the question $(x+3y)-(x+y)=2y=4$, or $y=2$, this becomes

$$x^4-40x^2=176841;$$

Conseq. $x^2=20+\sqrt{(176841+400)}=441$.

Hence $x=\sqrt{441}=21$; and $x-3y=15$, $x-y=19$ $x+y=23$, and $x+3y=27$, the numbers sought.

Ex. 21. Let x and y represent the two numbers; then by the question

$$\left.\begin{aligned} x^2+xy&=140 \\ y^2-xy&=\ \ 78 \end{aligned}\right\}$$

By addition $x^2+y^2=218$, or $x^2=218-y^2$,

But by the second equation $x=\dfrac{y^2-78}{y}$

$$\therefore\ 218-y^2=\left(\frac{y^2-78}{y}\right)^2$$

and $218y^2-y^4=y^4-156y^2+6084$,
which by transposing and dividing by 2 becomes

$$y^4-187y^2=-3042\,;$$

$$\therefore\ y^2=\frac{187}{2}+\sqrt{\left(\frac{187}{2}\right)^2-3042}=169.$$

Hence $y=13$, and $x=\sqrt{(218-y^2)}=7$, the numbers sought.

Ex. 22. Let $x=$ the number of hours' march of the first detachment, and y the miles per hour; then $x+1$ will be the hours of the second, and $y-\frac{1}{4}$ the miles per hour. Then by the question we have

$xy=39$, and $(x+1)(y-\frac{1}{4})=39$, or $xy-\frac{1}{4}x+y-\frac{1}{4}=39$;

Or, since $xy=39$, we have $-\frac{1}{4}x+y-\frac{1}{4}=0$,

and $\therefore\ 4y-x=1$.

Again, $x=\dfrac{39}{y}$; whence $4y-\dfrac{39}{y}=1$, or $4y^2-39=y$; hence $y^2-\dfrac{1}{4}y=\dfrac{39}{4}$, and $y=\dfrac{1}{8}+\sqrt{\left(\dfrac{1}{64}+\dfrac{39}{4}\right)}=\dfrac{26}{8}=3\frac{1}{4}$. Consequently $3\frac{1}{4}$ and 3 miles per hour are their rates of marching.

OF CUBIC EQUATIONS.

To exterminate the second term from a Cubic Equation.

Ex. 3. Given equation $x^3-6x^2=10$, or $x^3-6x^2-10=0$

Here $x=y+2$.

$$\text{Therefore}\left\{\begin{aligned} x^3&=y^3+6y^2+12y+\ 8 \\ -\ 6x^2&=\quad\ \ -6y^2-24y-24 \\ -10\ &=\qquad\qquad\qquad\quad\ -10 \end{aligned}\right.$$

Whence we shall have $y^3-12y-26=0$, or

$y^3-12y=26$, as required.

Ex. 4. Given equation $y^3-15y^2+81y-243=0$.

Here $y=x+5$,

$$\text{Therefore}\left\{\begin{array}{rl} y^3= & x^3+15x^2+\ \ 75x+125 \\ -15y^2= & \quad\ \ -15x^2-150x-375 \\ +81y= & \qquad\qquad\quad +\ \ 81x+405 \\ -243= & \qquad\qquad\qquad\qquad\ \ -243 \end{array}\right.$$

Whence we shall have $x^3+6x-88=0$, or

$x^3+6x=88$, as required.

Ex. 5. Given equation $x^3+\frac{3}{4}x^2+\frac{7}{8}x-\frac{9}{16}=0$.

Here $x=y-\frac{1}{4}$.

$$\text{Therefore}\left\{\begin{array}{rl} x^3= & y^3-\frac{3}{4}y^2+\frac{3}{16}y-\frac{1}{64} \\ +\frac{3}{4}x^2= & \quad\ +\frac{3}{4}y^2-\frac{3}{8}y+\frac{3}{64} \\ +\frac{7}{8}x= & \qquad\qquad\quad \frac{7}{8}y-\frac{7}{32} \\ -\frac{9}{16}\ \ = & \qquad\qquad\qquad\quad -\frac{9}{16} \end{array}\right.$$

Hence we have $y^3+\frac{11}{16}y-\frac{3}{4}=0$, or

$y^3+\frac{11}{16}y=\frac{3}{4}$ as required.

Ex. 6. Given equation $2x^3-3x^2+4x-5=0$, or

$$x^3-\frac{3}{2}x^2+2x-\frac{5}{2}=0.$$

Here $x=z+\frac{1}{2}$,

$$\text{Therefore}\left\{\begin{array}{rl} x^3 = & z^3+\frac{3}{2}z^2+\frac{3}{4}z+\frac{1}{8} \\ -\frac{3}{2}x^2 = & -\frac{3}{2}z^2-\frac{3}{2}z-\frac{3}{8} \\ +2x = & +2z+1 \\ -\frac{5}{2} = & -\frac{5}{2} \end{array}\right.$$

Whence we have $z^3+\frac{5}{4}z-\frac{7}{4}=0$, or

$$z^3+\frac{5}{4}z=\frac{7}{4} \text{ as required}$$

Where, as in the rest, the second term is wanting.

SOLUTION OF CUBIC EQUATIONS.

Ex. 1. Given $x^3+3x^2-6x=8$, to find x.

Here $x=y-1$,

$$\text{Therefore}\left\{\begin{array}{rl} x^3 = & y^3-3y^2+3y-1 \\ +3x^2 = & +3y^2-6y+3 \\ -6x = & -6y+6 \\ -8 = & -8 \end{array}\right.$$

Reduced equation $y^3-9y=0$, or $y^3=9y$

Consequently $y=0$, and, dividing by y, we have $y^2=9$, or $y=+3$, or -3; whence the three values of x are

$$\left.\begin{array}{l} x=y-1=0-1=-1 \\ x=y-1=3-1=2 \\ x=y-1=-3-1=-4 \end{array}\right\} \text{ as required.}$$

Ex. 2. Given $x^3+x^2=500$, to find x

First $x=z-\frac{1}{3}$,

$$\text{Therefore}\left\{\begin{array}{rl} x^3 = & z^3-z^2+\frac{1}{3}z-\frac{1}{27} \\ +\ x^2 = & z^2-\frac{2}{3}z+\frac{1}{9} \\ -500 = & -500 \end{array}\right.$$

Reduced equation $z^3 - \frac{1}{3}z - 499\frac{25}{27} = 0$, or

$$z^3 - \frac{1}{3}z = 499\tfrac{25}{27}$$

Whence we have $a = -\frac{1}{3}$ and $b = 499\frac{25}{27}$; consequently our formula

$$z = \sqrt[3]{\left\{\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}} + \sqrt[3]{\left\{\frac{b}{2} - \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}}$$

becomes

$$z = \sqrt[3]{\left\{\frac{499\frac{25}{27}}{2} + \sqrt{\left(\frac{(499\frac{25}{27})^2}{4} - \frac{1}{729}\right)}\right\}} +$$
$$\sqrt[3]{\left\{\frac{499\frac{25}{27}}{2} - \sqrt{\left(\frac{(499\frac{25}{27})^2}{4} - \frac{1}{729}\right)}\right\}},$$

Which last expression, being reduced, gives

$$z = \sqrt[3]{\left\{249\cdot962963 + \sqrt{\left(249\cdot962963^2 - \frac{1}{729}\right)}\right\}} +$$
$$\sqrt[3]{\left\{249\cdot962963 - \sqrt{\left(249\cdot962963^2 - \frac{1}{729}\right)}\right\}}$$
$$= \sqrt[3]{499\cdot925923} + \sqrt[3]{\cdot0000027}$$
$$= 7\cdot9366 + \cdot0139$$
$$= 7\cdot9505$$

Whence $x = z - \frac{1}{3} = 7\cdot9505 - \cdot3333 = 7\cdot6172$

Ex. 3. Given $x^3 + 12x = 20$.

Here a being $= 12$, and $b = 20$, we shall have by the formula

$$x = \sqrt[3]{\left\{\frac{20}{2} + \sqrt{\left(\frac{20^2}{4} + \frac{12^3}{27}\right)}\right\}} + \sqrt[3]{\left\{\frac{20}{2} - \sqrt{\left(\frac{20^2}{4} + \frac{12^3}{27}\right)}\right\}},$$
$$= \sqrt[3]{\{10 + \sqrt{(100 + 64)}\}} + \sqrt[3]{\{10 - \sqrt{(100 + 64)}\}}$$
$$= \sqrt[3]{\{10 + \sqrt{164}\}} + \sqrt[3]{\{10 - \sqrt{164}\}}$$
$$= \sqrt[3]{\{10 + \sqrt{12\cdot80625}\}} + \sqrt[3]{\{10 - 12\cdot80625\}}$$
$$= \sqrt[3]{(22\cdot80625)} - \sqrt[3]{(2\cdot80625)}$$
$$= 2\cdot83586 - 1\cdot41051$$
$$= 1\cdot42535 \quad \text{Answer.}$$

Ex. 4. The given equation $x^3 - 6x = 6$, being in its proper reduced form, we have

$$x = \sqrt[3]{\{3 + \sqrt{(9-8)}\}} + \sqrt[3]{\{3 - \sqrt{(9-8)}\}},$$
$$= \sqrt[3]{(3+1)} + \sqrt[3]{(3-1)}, \text{ or } x = \sqrt[3]{4} + \sqrt[3]{2} \quad \text{Ans.}$$

Ex. 5. The given equation $x^3+9x=6$ being here also in its reduced form, we shall have

$$x=\sqrt[3]{\{3+\sqrt{(9+27)}\}}+\sqrt[3]{\{3-\sqrt{(9+27)}\}},$$
$$=\sqrt[3]{(3+6)}+\sqrt[3]{(3-6)}, \text{ or}$$
$$=\sqrt[3]{9}+\sqrt[3]{-3}=\sqrt[3]{9}-\sqrt[3]{3} \quad \text{Ans.}$$

Ex. 6. Given $x^3-22x=24$, to find x.

Here, by the formula, we have

$$x=\sqrt[3]{\{12+\sqrt{(144-\frac{10648}{27})}\}}+$$
$$\sqrt[3]{\{12-\sqrt{(144-\frac{10648}{27})}\}},$$

Or by reducing and simplifying the expression

$$x=\sqrt[3]{\{12+\frac{1}{3}\sqrt{(144\times27-10648)}\}}+$$
$$\sqrt[3]{\{12-\frac{1}{3}\sqrt{(144\times27-10648)}\}}$$
$$=\sqrt[3]{\{12+\frac{1}{3}\sqrt{(3888-10648)}\}}+$$
$$\sqrt[3]{\{12-\frac{1}{3}\sqrt{(3888-10648)}\}}$$
$$=\sqrt[3]{(12+\frac{1}{3}\sqrt{-6760})}+\sqrt[3]{(12-\frac{1}{3}\sqrt{-6760})}.*$$

Ex. 7. Given equation $x^3-17x^2+54x=350$, to find x.

This equation, by exterminating the second term, becomes

$$z^3-42\frac{1}{3}z=407\frac{25}{27},$$

Whence, by the formula, we shall have

$$z=\sqrt[3]{\{203\frac{26}{27}+\sqrt{[(203\frac{26}{27})^2-(14\frac{1}{9})^3]}\}}$$
$$+\sqrt[3]{\{203\frac{26}{27}-\sqrt{[(203\frac{26}{27})^2-(14\frac{1}{9})^3]}\}}$$

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* This Question falling under the irreducible case, can only be resolved by a table of sines, or by infinite series; for the method of doing which see my *Treatise on Algebra*, 2 vols. 8vo. 2d edit., 1820.

Or, by reducing and simplifying the expression,

$$z=\frac{1}{3}\sqrt[3]{\{5507+\sqrt{(5507^2-127^3)}\}}+$$
$$\frac{1}{3}\sqrt[3]{\{5507-\sqrt{(5507^2-127^3)}\}},$$
$$=\frac{1}{3}\sqrt[3]{\{5507+\sqrt{28278666}\}}+$$
$$\frac{1}{3}\sqrt[3]{\{5507-\sqrt{28278666}\}}$$
$$=\frac{1}{3}(\sqrt[3]{5507+5317\cdot7689})+$$
$$\frac{1}{3}(\sqrt[3]{5507-5317\cdot7689})$$
$$=\frac{1}{3}\sqrt[3]{10824\cdot7689}+\frac{1}{3}\sqrt[3]{189\cdot2311}=9\cdot28762.$$

Consequently $x=z+\frac{17}{3}=14\cdot95429$ Ans.

OF BIQUADRATIC EQUATIONS.

Ex. 2. Here the given equation, viz.

$$x^4-55x^2-30x+504=0,$$

being of the proper form for solution, we have, by the first rule, $b=-55$, $c=-30$, and $d=504$.

The cubic or reduced equation,

$$z^3-(\frac{1}{12}b^2+d)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd$$

becomes

$$z^3-756\tfrac{1}{12}z=7811\tfrac{107}{108}$$

Where the root of this cubic being fo $=31\cdot66666$, or $31\frac{2}{3}$, our two quadratics are

$$x^2+\sqrt{\{2(31\frac{2}{3}+\frac{55}{3})\}}x=-(31\frac{2}{3}-\frac{55}{6})$$
$$+\sqrt{\{(31\frac{2}{3}-\frac{55}{6})^2-504\}}$$

$$x^2-\sqrt{\{2(31\frac{2}{3}+\frac{55}{3})\}}x=-(31\frac{2}{3}-\frac{55}{6})$$
$$-\sqrt{\{(31\frac{2}{3}-\frac{55}{6})^2-504\}}.$$

Which reduce to

$$x^2+10x=-\frac{45}{2}\pm\frac{3}{2}$$
$$x^2-10x=-\frac{45}{2}\mp\frac{3}{2}.$$

As the product of the reduced quadratics must give the proposed biquadratic, we have hence the criterion of judging which sign is to be used. It will be found by taking the lower sign that this property will be fulfilled, so that our two quadratics are

$$x^2+10x=-24, \text{ and } x^2-10x=-21.$$

The first gives $x=-5\pm\sqrt{(25-24)}=-4$ or -6.
And the second $x=\ \ 5\pm\sqrt{(25-21)}=\ \ 7$ or $\ \ 3$.
Therefore the four roots are 3, 7, -4, and -6.

Ex. 3. Given equation $x^4+2x^3-7x^2-8x=-12$.
First, in order to exterminate the second term, we have $x=z-\frac{1}{2}$; whence

$$\begin{array}{rl} x^4= & z^4-2z^3+\frac{3}{2}z^2-\frac{1}{2}z+\frac{1}{16} \\ +2x^3= & \qquad +2z^3-3z^2+\frac{3}{2}z-\frac{1}{4} \\ -7x^2= & \qquad\qquad -7z^2+7z-\frac{7}{4} \\ -8x= & \qquad\qquad\qquad -8z+\ 4 \\ +12= & \qquad\qquad\qquad\qquad +12 \\ \hline \end{array}$$

The reduced equation $z^4-8\frac{1}{2}z^2+14\frac{1}{16}=0$, or

$$z^4-8\frac{1}{2}z^2=-14\frac{1}{16}$$

Whence $z^2=4\frac{1}{4}\pm\sqrt{(\frac{289}{16}-\frac{225}{16})}=\frac{17}{4}\pm\frac{8}{4}=\frac{25}{4}$, or $\frac{9}{4}$,

Consequently $z=\pm\sqrt{\frac{25}{4}}=+\frac{5}{2}$, or $-\frac{5}{2}$; also

$$z=\pm\sqrt{\frac{9}{4}}=+\frac{3}{2}, \text{ or } -\frac{3}{2};$$

And therefore $x=z-\frac{1}{2}$ has the four following values, viz.

$$x=\frac{5}{2}-\frac{1}{2}=\frac{4}{2}=2$$

$$x=-\frac{5}{2}-\frac{1}{2}=-\frac{6}{2}= \quad .$$

$$x=\frac{3}{2}-\frac{1}{2}=\frac{2}{2}= \quad 1$$

$$x=-\frac{3}{2}-\frac{1}{2}=-\frac{4}{2}=-2$$

Ex. 4. Given $x^4-8x^3+14x^2+4x=8$ to find x.

First $x=y+2$

$$\begin{array}{lrl}
\text{Whence} & x^4= & y^4+8y^3+24y^2+32y+16 \\
 & -\ 8x^3= & -8y^3-48y^2-96y-64 \\
 & +14x^2= & \qquad\quad +14y^2+56y+56 \\
 & +\ 4x= & \qquad\qquad\qquad\quad 4y+\ 8 \\
 & -\ 8= & \qquad\qquad\qquad\qquad\ -\ 8 \\
\hline
\end{array}$$

Reduced equation $y^4-10y^2-4y+8=0$

From which we obtain the following cubic,

$$z^2-(\frac{100}{12}+8)z=-\frac{1000}{108}+\frac{16}{8}+\frac{80}{3}, \text{ or}$$

$$z^3-16\frac{1}{3}z=19\frac{11}{27}.$$

The root of which cubic is $z=-1\cdot3333$, or $-\frac{4}{3}$.

Therefore the two quadratic equations are

$$y^2+\sqrt{\{2(-\frac{4}{3}+\frac{10}{3})\}}y=-(-\frac{4}{3}-\frac{10}{6})$$
$$+\sqrt{\{(-\frac{4}{3}-\frac{10}{6})^2-8\}}$$

$$y^2-\sqrt{\{2(-\frac{4}{3}+\frac{10}{3})\}}y=-(-\frac{4}{3}-\frac{10}{6})$$
$$-\sqrt{\{(-\frac{4}{3}-\frac{10}{6})^2-8\}}$$

Which reduce to

$$y^2+2y=3\pm 1$$
$$y^2-2y=3\mp 1$$

Attending to the criterion noticed in the solution to our second Example, we shall learn that we must use the lower sign: hence the reduced quadratics are

$$y^2+2y=2, \text{ and } y^2-2y=4.$$

Whence the four values of y, are

$$-1+\sqrt{3},\ -1-\sqrt{3},\ 1+\sqrt{5},\ 1-\sqrt{5}.$$

And since $x=y+2$, we have the four following values of x, viz.

$$1+\sqrt{3},\ 1-\sqrt{3},\ 3+\sqrt{5},\ 3-\sqrt{5}$$

which are the four roots of the proposed equation.

Ex. 5. Here the given equation, viz.

$$x^4-17x^2-20x-6=0$$

is already in the proper form for solution, we have $b=-17$, $c=-20$, and $d=-6$; whence our cubic is

$$z^3-(\frac{17^2}{12}-6)z=\frac{-17^3}{108}+\frac{20^2}{8}-\frac{17\times 6}{3}, \text{ or}$$

$$z^3-18\frac{1}{12}z=-29\frac{53}{108},$$

The root of which is $z=2\cdot 3333$, or $z=2\frac{1}{3}$

Whence our quadratics are

$$x^2+\sqrt{\{2(\frac{7}{3}+\frac{17}{3})\}}x=-(\frac{7}{3}-\frac{17}{6})$$
$$+\sqrt{\{(\frac{7}{3}-\frac{17}{6})^2+6\}}$$

$$x^2-\sqrt{\{2(\frac{7}{3}+\frac{17}{3})\}}x=-(\frac{7}{3}-\frac{17}{6})$$
$$-\sqrt{\{(\frac{7}{3}-\frac{17}{6})^2+6\}}$$

Which reduce to

$$x^2+4x=\frac{1}{2}\pm 2\frac{1}{2}$$

$$x^2-4x=\frac{1}{2}\mp 2\frac{1}{2}$$

The criterion noticed in the solution to our second Example teaches us that we must use the lower sign: hence the reduced quadratics are

$$x^2+4x=-2, \text{ and } x^2-4x=3.$$

Therefore the four roots, or values of x, are

$$x=-2+\sqrt{2},\ x=-2-\sqrt{2}$$
$$x=+2+\sqrt{7},\ x=+2-\sqrt{7}, \text{ as required.}$$

Ex. 6. Given equation $x^4-27x^3+162x^2+356x-1200=0$.

First $x=z+\frac{27}{4}$; then

$$\begin{array}{rl}
x^4= & z^4+27z^3+\frac{2187}{8}z^2+\frac{19683}{16}z+\frac{531441}{256} \\
-27x^3= & -27z^3-\frac{2187}{4}z^2-\frac{59049}{16}z-\frac{531441}{64} \\
+162x^2= & +162z^2+2187z+\frac{59049}{8} \\
+356x= & +356z+2403 \\
-1200= & -1200
\end{array}$$

Reduced equation $z^4-111\frac{3}{8}z^2+82\frac{5}{8}z+2356\frac{77}{256}=0$

From which we derive the following cubic, viz

$$y^3-3390y=75539,$$

The root of which is $67\cdot188435$.

Whence our quadratics are

$$z^2+\sqrt{\{2(67\cdot188435+37\cdot125)\}}z=-(67\cdot188435-18\cdot5625)+\sqrt{\left\{(67\cdot188435-18\cdot5625)^2-2356\frac{77}{256}\right\}}$$

$$z^2-\sqrt{\{2(67\cdot188435+37\cdot125)\}}z=-(67\cdot188435-18\cdot5625)-\sqrt{\left\{(67\cdot188435-18\cdot5625)^2-2356\frac{77}{256}\right\}},$$

Wh ch reduce to

$$z^2+14\cdot443922z=-48\cdot625935\pm2\cdot860205$$
$$z^2-14\cdot443922z=-48\cdot625935\mp2\cdot860205$$

Taking the upper signs, suggested by the criterion stated in the solution to our second Example, these quadratics become

$$z^2+14\cdot443922z=-45\cdot76573$$
$$z^2-14\cdot443922z=\quad 51\cdot48614$$

Whence the four values of z are

$$-4\cdot6939,\ -9\cdot75,\ 6\cdot4031,\ 8\cdot0409,$$

And as $x=z+6\cdot75$, the values of x are

$$2\cdot0561,\ -3,\ 13\cdot1531,\ 14\cdot7909.$$

Ex. 7. Given $x^4-12x^2+12x-3=0$, to find x.

Here the equation is in its proper form for solution, having $b=-12$, $c=12$, and $d=-3$; our cubic is

$$z^3-(12-3)z=-\frac{1728}{108}+\frac{144}{8}-12, \text{ or}$$
$$z^3-9z=-10$$

Where we have, from inspection, $z=2$; whence the following quadratics,

$$x^2+\sqrt{\{2(2+4)\}}x=-(2-2)+\sqrt{\{(2-2)^2+3\}}$$
$$x^2-\sqrt{\{2(2+4)\}}x=-(2-2)-\sqrt{\{(2-2)^2+3\}}$$

Or $\quad x^2+\sqrt{12}x=\quad\sqrt{3}$

$\quad\quad x^2-\sqrt{12}x=-\sqrt{3}$

Whence $x=-\frac{1}{2}\sqrt{12}\pm\sqrt{(3+\sqrt{3})}$

$\quad\quad x=+\frac{1}{2}\sqrt{12}\pm\sqrt{(3-\sqrt{3})}$

Therefore the four values of x in numbers are

$$x=-3\cdot907377;\ x=\cdot443279$$
$$x=\quad 2\cdot858084;\ x=\cdot606018.$$

RESOLUTION OF EQUATIONS,

BY APPROXIMATION.

Ex. 2. Given equation $x^2+20x=100$.

Here a few trials show the root to be nearly $4\cdot1$; let therefore $\quad x=4.1+z$;

Then $\left.\begin{array}{l} x^2=16\cdot81+8\cdot2z+z^2 \\ 20x=82\quad\ +20z \end{array}\right\}=100.$

Therefore by rejecting z^2, we have

$$98\cdot81+28\cdot2z=100, \text{ or}$$
$$z=\frac{1\cdot19}{28\cdot2}=\cdot042.$$

And consequently $x=4\cdot1+z=4\cdot1+\cdot042=4\cdot142$;

Whence, by repeating the operation, and assuming $x=4\cdot142+z$, four other figures may be obtained which gives the value of $x=4\cdot1421356$, as required.

Ex. 3. Given $x^3+9x^2+4x=80$.

Here, by trial, we find the root nearly $=2$; assuming, therefore, $x=2+z$, we have

$$\left.\begin{array}{l} x^3=\ 8+12z+6z+z^3 \\ 9x^2=36+36z+9z^2 \\ 4x=\ 8+\ 4z \end{array}\right\}=80$$

Whence, by rejecting the second and third powers of z, we have

$$52+52z=80,\text{ or } z=\frac{28}{52}=\cdot5$$

And therefore $x=2\cdot5=$ the root *nearly*.

Assume now $x=2\cdot5+z$, and we have

$$\left.\begin{array}{llll} x^3=15\cdot625 & +18\cdot75z & +7\cdot5z^2 & +z^3 \\ 9x^2=56\cdot25 & +45z & +9z^2 & \\ 4x=10\cdot & +\ 4z & & \end{array}\right\}=80$$

Whence, rejecting the high powers of z,

$$81\cdot875+67\cdot75z=80,\text{ or}$$

$$z=-\frac{1\cdot875}{67\cdot75}=-\cdot0277$$

Where $x=2\cdot5-\cdot0277=2\cdot472$ *nearly*; and if we were again to assume $x=2\cdot472+z$, we should get four other decimals, which would give $x=2\cdot4721359$.

Remark. In the preceding examples we have involved every power of x completely; but it is obvious, that as the higher powers of z are always rejected, it is unnecessary to carry the expansions beyond the second term, as shown in the following solution.

Ex. 4. Given equation $x^4-38x^3+210x^2+538x+289=0$.

Here, since the second term is equal to all the other terms of the equation, it is obvious that x is less than 38; and by a few trials we find it to be nearly 30; let therefore $x=30+z$; then, reserving only the first two terms of each expansion, we have

$$\left.\begin{array}{rl} x^4 = & 810000+108000z+\&c. \\ -\ 38x^3 = & -1026000-102600z-\&c. \\ +210x^2 = & 189000+\ 12600z+\&c. \\ +538x = & 16140+\quad 538z+\&c. \\ +289\ = & 289 \end{array}\right\}=0$$

Whence $-10571+18538z=0$, or $z=\dfrac{10571}{18538}=\cdot 5$,

And $x=30\cdot 5$ *nearly*. Assume therefore again $x=30\cdot 5+z$, and we shall have (rejecting the decimals as inconsiderable)

$$\left.\begin{array}{rl} x^4 = & 865365+113490z \\ -\ 38x^3 = & -1078160-106049z \\ +210x^2 = & 195352+\ 12810z \\ +538x = & 16409+\quad 538z \\ +289\ = & 289 \end{array}\right\}=0$$

$$\text{Whence}\qquad -745+\ 20789z\ =0$$

Or $z=\dfrac{745}{20789}=\cdot 037$; $\therefore\ x=30\cdot 53$ *nearly*;

And by assuming again $x=30\cdot 53+z$, a still nearer approximation may be obtained.

But in equations of this kind, in which the coefficients are large with regard to the root, the approximations are always very slow, adding no more than one new figure at each operation; whereas, when the coefficients are small with respect to the root, each operation doubles the number of figures last obtained.

Ex. 5. Given $x^5+6x^4-10x^3-112x^2-207x+110=0$, to find x.

Here $x=4$ *nearly*; assume, therefore, $x=4+z$; then

$$\left.\begin{array}{rl} x^5 = & 1024+1280z+\&c. \\ +\ \ 6x^4 = & 1536+1536z+\&c. \\ -\ \ 10x^3 = & -\ \ 640-\ \ 480z-\&c. \\ -112x^2 = & -1792-\ \ 896z-\&c. \\ -207x = & -\ \ 828-\ \ 207z-\&c. \\ +110\ = & +\ \ 110 \end{array}\right\}=0$$

And $-590+1233z=0$, or $z=\dfrac{590}{1233}=\cdot 47$.

Whence $x=4\cdot47$ *nearly*. And, by assuming $x=4\cdot47+z$, another approximation may be obtained, till at last we find $x=4\cdot46410161$.

But it would be useless to go through the entire operation in this place.

OF APPROXIMATION

BY POSITION.

EXAMPLES FOR PRACTICE.

Ex. 1. Given $x^3+10x^2+5x=2600$.

Here it is soon discovered that x is little more than 11; let us assume, therefore, $x=11\cdot0$ and $x=11\cdot1$;

Then by the rule

$11\cdot1$	=	x	=	$11\cdot0$
$1367\cdot631$	=	x^3	=	1331
$1232\cdot1$	=	$+10x^2$	=	1210
$55\cdot5$	=	$+\ 5x$	=	55
$2655\cdot231$		Results		2596

Therefore

$2655\cdot231$	$11\cdot1$	2600
2596	$11\cdot0$	2596
Or $59\cdot231$:	$0\cdot1$ ::	$4 : \cdot00675$

Whence $x=11\cdot0+\cdot00675=11\cdot00675$ Ans.

Remark. In this example one of our suppositions approached so near the truth, that we have been enabled to obtain seven figures true, in a single operation; which is a degree of approximation very seldom acquired with so little labour.

Ex. 2. Given $2x^4-16x^3+40x^2-30x+1=0$.

Here x is nearly $=1$; assume, therefore,

$$2 \quad = \quad x \quad = \quad 1$$

Then, by the rule, we have

32	=	$2x^4$	=	2
-128	=	$-16x^3$	=	-16
$+160$	=	$+40x^2$	=	$+40$
-60	=	$-30x$	=	-30
$+1$		$+1$	=	$+1$
$+5$		Results		-3

Therefore

	$+5$		2		0	
	-3		1		-3	
Or	8	:	1	::	3	: ·3

Whence $x=1\cdot 3$ *nearly;* assume, therefore, now

	1·3	=	x	=	1·2
Then	5·7122	=	$2x^4$	=	$+$ 4·1472
	$-35\cdot 152$	=	$-16x^3$	=	$-27\cdot 648$
	$+67\cdot 6$	=	$+40x^2$	=	$+57\cdot 6$
	$-39\cdot$	=	$-30x$	=	$-36\cdot$
	$+1\cdot$	=	$+1$	=	$+1\cdot$
	$+\cdot 1602$		Results		$-\cdot 9008$

Therefore

	$+\cdot 1602$		1·3		0·	
	$-\cdot 9008$		1·2		$-\cdot 9008$	
Or	1·061	:	·1	::	·9008	: ·085

Hence $x=1\cdot 2+\cdot 085=1\cdot 285$ *nearly;* and by repeating the operation we shall obtain the still nearer approximation 1·284724.

Ex. 3. Given $x^5+2x^4+3x^3+4x^2+5x=54321$.

Here we find the value of x to be between 8 and 9; assume, therefore,

	8	=	x	=	9
Then	32768	=	x^5	=	59049
	8192	=	$2x^4$	=	13122
	1536	=	$3x^3$	=	2187
	256	=	$4x^2$	=	324
	40	=	$5x$	=	45
	42792		Results		74727

Therefore

$$\begin{array}{ccccccc} 74727 & & 9 & & 54321 & & \\ 42792 & & 8 & & 42792 & & \\ \hline 31935 & : & 1 & :: & 11529 & : & \cdot 3 \end{array}$$

Whence $x=8 \cdot 3$ *nearly*. And, by assuming $x=8 \cdot 3$ and $8 \cdot 4$, another approximation will be obtained; but the successive corrections are very small, and would occupy more room than can be devoted to a single example.

Ex. 4. Given $\sqrt[3]{(7x^3+4x^2)}+\sqrt{(20x^2-10x)}=28$.

Assuming here $x=4$ and $x=5$, we have

$$\begin{array}{ccc} 4 = & x & = 5 \\ \hline 8 & =\sqrt[3]{(7x^3+4x^2)}= & 9 \cdot 91596 \\ 16 \cdot 7332 & =\sqrt{(20x^2-10x)}= & 21 \cdot 21320 \\ \hline 24 \cdot 7332 & \text{Results} & 31 \cdot 12916 \end{array}$$

Therefore

$$\begin{array}{ccccccc} 31 \cdot 12916 & & 5 & & 28 & & \\ 24 \cdot 7332 & & 4 & & 24 \cdot 7332 & & \\ \hline 6 \cdot 39596 & : & 1 & :: & 3 \cdot 2668 & : & \cdot 51 \end{array}$$

Whence we shall have $x=4 \cdot 51$ *nearly*.

And by repeating the operation $x=4 \cdot 510661$.

Ex. 5. Given $\sqrt{\{144x^2-(x^2+20)^2\}}+\sqrt{\{196x^2-(x^2+24)^2\}}=114$.

Assuming $x=7$, and $x=8$, we have

$$\begin{array}{ccc} 7 = & x & = 8 \\ \hline 47 \cdot 9062 & =\sqrt{\{144x^2-(x^2+20)^2\}}= & 46 \cdot 4758 \\ 65 \cdot 3835 & =\sqrt{\{196x^2-(x^2+24)^2\}}= & 69 \cdot 2820 \\ \hline 113 \cdot 2897 & \text{Results} & 115 \cdot 7578 \end{array}$$

Therefore

$$\begin{array}{ccccccc} 115 \cdot 7578 & & 8 & & 114 & & \\ 113 \cdot 2897 & & 7 & & 113 \cdot 2897 & & \\ \hline 2 \cdot 4681 & : & 1 & :: & \cdot 7103 & : & \cdot 2 \end{array}$$

Whence $x=7 \cdot 2$ nearly.

And as $x=7 \cdot 2$ is found too great; let us therefore take $x=7 \cdot 1$ and $7 \cdot 2$, and we shall have

$$
\begin{array}{ccc}
7\cdot 1 & =x= & 7\cdot 2 \\
\hline
47\cdot 9737=\surd\{144x^2-(x^2+20)^2\}=47\cdot 9997 \\
65\cdot 9053=\surd\{196x^2-(x^2+24)^2\}=66\cdot 3998 \\
\hline
113\cdot 8790 & \text{Results} & 114\cdot 3995
\end{array}
$$

$$
\begin{array}{ccc}
114\cdot 3995 & 7\cdot 2 & 114 \\
113\cdot 8790 & 7\cdot 1 & 113\cdot 8790 \\
\hline
\cdot 5205 \;:\; & \cdot 1 \;::\; & \cdot 121 : \cdot 023
\end{array}
$$

Whence $x=7\cdot 1+\cdot 023=7\cdot 123$ *nearly.*

And if, for a new operation, there be taken $x=7\cdot 123$ and $7\cdot 124$, we shall have $x=7\cdot 123883$.

And a further assumption of this kind will about double the number of decimals.

EXPONENTIAL EQUATIONS.

Ex. 2. Given $x^x=2000$ to find x.

Here we soon find that x is nearly$=5$; let us, therefore, assume $x=4\cdot 8$, and $x=4\cdot 9$.

Then the log. $2000=3\cdot 3010300$; and

$$
\begin{array}{lrcr}
\text{Log. } 4\cdot 8= & 0\cdot 6812412 & & \text{Log. } 4\cdot 9=0\cdot 6901961 \\
\text{Mult. by} & 4\cdot 8 & & 4\cdot 9 \\
\hline
& 54499296 & & 62117649 \\
& 27249648 & & 27607844 \\
\hline
& 3\cdot 26995776 & \text{Results} & 3\cdot 38196089
\end{array}
$$

Therefore

$$
\begin{array}{ccc}
3\cdot 38196 & 4\cdot 9 & 3\cdot 30103 \\
3\cdot 26995 & 4\cdot 8 & 3\cdot 26995 \\
\hline
\cdot 11201 \;:\; & \cdot 1 \;::\; & \cdot 03108 : \cdot 027
\end{array}
$$

Whence $x=4\cdot 8+\cdot 027=4\cdot 827$ *nearly.*

And, repeating the operation by assuming $x=4\cdot 827$ and $x=4\cdot 828$, we shall obtain $x=4\cdot 82782263$.

Ex. 3. Here $(6x)^x=96$; and our formula must therefore be $x\times(\text{log. } 6+\text{log. } x)=\text{log. } 96$.

Where log. $96=1\cdot 9822712$,

And x is obviously nearly $=2$; assume, therefore,

$x=1\cdot8$; and $x=1\cdot9$; then

$$\begin{array}{lr}
\text{Log. } 6 = 0\cdot77815 & \text{Log. } 6 = 0\cdot77815 \\
\text{Log. } 1\cdot8 = 0\cdot25527 & \text{Log. } 1\cdot9 = 0\cdot27875 \\
\hline
1\cdot03342 & 1\cdot05690 \\
1\cdot8 & 1\cdot9 \\
\hline
826736 & 951210 \\
103342 & 105690 \\
\hline
1\cdot860156 \quad \text{Results} & 2\cdot008110
\end{array}$$

Therefore

$$\begin{array}{ccccccc}
2\cdot00811 & & 1\cdot9 & & 1\cdot98227 & & \\
1\cdot86015 & & 1\cdot8 & & 1\cdot86015 & & \\
\hline
\cdot14796 & : & \cdot1 & :: & \cdot12212 & : & \cdot0826,
\end{array}$$

Whence $x=1\cdot8+\cdot0826=1\cdot8826$ nearly.

By another operation, we shall have $x=1\cdot8826432$.

Ex. 4. Given equation $x^x=123456789$.

Here, after a few trials, or from inspection in a table of powers, we find x is between 8 and 9, but nearer the latter than the former. Assume, therefore, $x=8\cdot6$ and $x=8\cdot7$.

Then log. $123456789=8\cdot0915150$.

$$\begin{array}{lr}
\text{Log. } 8\cdot6 = 0\cdot9344985 & \text{Log. } 8\cdot7 = 0\cdot9395193 \\
\text{Mult. by} \quad 8\cdot6 & 8\cdot7 \\
\hline
56069910 & 65766351 \\
74759880 & 75161544 \\
\hline
8\cdot03668710 \quad \text{Results} & 8\cdot17381791
\end{array}$$

Therefore

$$\begin{array}{ccccccc}
8\cdot17381 & & 8\cdot7 & & 8\cdot09151 & & \\
8\cdot03668 & & 8\cdot6 & & 8\cdot03668 & & \\
\hline
\cdot13713 & : & \cdot1 & :: & \cdot05483 & : & \cdot040
\end{array}$$

Whence $x=8\cdot6+\cdot040=8\cdot640$ nearly;

And repeating the operation, by assuming x equal to $8\cdot640$ and $8\cdot641$, x is found $=8\cdot6400268$.

Ex. 5. Given $x^x - x = (2x - x^x)^{\frac{1}{x}}$.

This will be more convenient under the form

$$(x^x - x)^x = 2x - x^x.$$

Now, in order to find a first approximate assumption, it may be observed that $2x$ must be greater than x^x, or 2 greater than x^{x-1}.

The same will also be obvious if we put the equation under the form $\dfrac{(x^x - x)^x}{x} + x^{x-1} = 2$.

Since then x is less than 2, but nearly equal to that number, let us assume $x = 1\cdot 8$; then $x^x = 2\cdot 8806$, and

$$\begin{array}{rcl}
\dfrac{(x^x - x)^x}{x} & = & 0\cdot 6387 \\
\text{Also } x^{x-1} & = & 1\cdot 6004 \\
\hline
\text{For the 1st result} & & 2\cdot 2391 \\
\hline
\text{And, when } x = 1\cdot 7, \text{ then } x^x = & & 2\cdot 4647 \\
\text{And } \dfrac{(x^x - x)^x}{x} & = & 0\cdot 3728 \\
\text{Also } x^{x-1} & = & 1\cdot 4498 \\
\hline
\text{For the 2d result} & & 1\cdot 8226 \\
\hline
\end{array}$$

Therefore, by the rule,

$$\begin{array}{lccccccc}
 & 2\cdot 2391 & & 1\cdot 8 & & 2\cdot 0000 & & \\
 & 1\cdot 8226 & & 1\cdot 7 & & 1\cdot 8226 & & \\
\hline
\text{As} & \cdot 4165 & : & \cdot 1 & :: & \cdot 1774 & : & \cdot 04;
\end{array}$$

Whence $x = 1\cdot 7 + \cdot 04 = 1\cdot 74$ nearly.

And, repeating the operation with the assumption $x = 1\cdot 74$ and $x = 1\cdot 75$, we find $x = 1\cdot 747933$.

Also, by another assumption, a still nearer approximate value of x may be determined; and so on, to any degree of accuracy required.

BINOMIAL THEOREM.

Ex. 5. Here the proposed binomial being $\sqrt{(1+1)}$, or $(1+1)^{\frac{1}{2}}$, we have $P = 1$, $Q = \frac{1}{1} = 1$, $\frac{m}{n} = \frac{1}{2}$

Whence

$$\text{P}^{\frac{m}{n}} = 1^{\frac{m}{n}} = 1^{\frac{1}{2}} = 1 = \text{A},$$

$$\frac{m}{n}\text{AQ} = \frac{1}{2} \times \frac{1}{1} \times \frac{1}{1} = \frac{1}{2} = \text{B},$$

$$\frac{m-n}{2n}\text{BQ} = \frac{1-2}{4} \times \frac{1}{2} \times \frac{1}{1} = \frac{-1}{2.4} = \text{C}$$

$$\frac{m-2n}{3n}\text{CQ} = \frac{1-4}{6} \times \frac{-1}{2.4} \times \frac{1}{1} = \frac{1.3}{2.4.6} = \text{D},$$

$$\frac{m-3n}{4n}\text{DQ} = \frac{1-6}{8} \times \frac{1.3}{2.4.6} \times \frac{1}{1} = \frac{-1.3.5}{2.4.6.8} = \text{E},$$

Where the law of continuation is sufficiently obvious; and therefore we have $\sqrt{(1+1)}$, or $\sqrt{2} =$

$$1 + \frac{1}{2} - \frac{1}{2.4} + \frac{1.3}{2.4.6} - \frac{1.3.5}{2.4.6.8} + \frac{1.3.5.7}{2.4.6.8.10} - \&\text{c.}$$

Ex. 6. Here we have to convert $(8-1)^{\frac{1}{3}}$, or $(2^3-1)^{\frac{1}{3}}$ into a series. Make $2 = a$; then it becomes $(a^3-1)^{\frac{1}{3}}$; where $\text{P} = a^3$, $\text{Q} = \frac{-1}{a^3}$, and $\frac{m}{n} = \frac{1}{3}$; whence

$$\text{P}^{\frac{m}{n}} = (a^3)^{\frac{m}{n}} = (a^3)^{\frac{1}{3}} = a = \text{A},$$

$$\frac{m}{n}\text{AQ} = \frac{1}{3} \times \frac{a}{1} \times \frac{-1}{a^3} = \frac{-1}{3a^2} = \text{B},$$

$$\frac{m-n}{2n}\text{BQ} = \frac{1-3}{6} \times \frac{-1}{3a^2} \times \frac{-1}{a^3} = \frac{-1.2}{3.6a^5} = \text{C},$$

$$\frac{m-2n}{3n}\text{CQ} = \frac{1-6}{9} \times \frac{-1.2}{3.6a^5} \times \frac{-1}{a^3} = \frac{-1.2.5}{3.6.9a^8} = \text{D},$$

Therefore $\sqrt[3]{(a^3-1)} =$

$$a - \frac{1}{3.a^2} - \frac{1.2}{3.6a^5} - \frac{1.2.5}{3.6.9a^8} - \frac{1.2.5.8}{3.6.9.12a^{11}} - \&\text{c.}$$

Or, substituting 2 and its powers for a, we have $\sqrt[3]{(8-1)}$, or $\sqrt[3]{7} =$

$$2 - \frac{1}{3.2^2} - \frac{1}{3.6.2^4} - \frac{1.5}{3.6.9.2^7} - \frac{1.5.8}{3.6.9.12.2^{10}} - \&\text{c}$$

Ex. 7. Here $(243-3)^{\frac{1}{5}}=(3^5-3)^{\frac{1}{5}}=(a^5-3)^{\frac{1}{5}}$ by writing $a=3$; also $P=a^5$, $Q=\frac{3}{a^5}$ and $\frac{m}{n}=\frac{1}{5}$; whence

$$P^{\frac{m}{n}}=(a^5)^{\frac{m}{n}}=(a^5)^{\frac{1}{5}}=a=A,$$

$$\frac{m}{n}AQ=\frac{1}{5}\times\frac{a}{1}\times\frac{-3}{a^5}=\frac{-3}{5a^4}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-5}{10}\times\frac{-3}{5a^4}\times\frac{-3}{a^5}=\frac{-4.3^2}{5.10a^9}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-10}{15}\times\frac{-4.3^2}{5.10a^9}\times\frac{-3}{a^5}=\frac{-4.9.3^3}{5.10.15a^{14}}=D.$$

Wherefore, by writing 3 for a, and cancelling the like powers of 3, we have $\sqrt[5]{(243-3)}$, or $\sqrt[5]{240}=$

$$3-\frac{1}{5.3^3}-\frac{4}{5.10.3^7}-\frac{4.9}{5.10.15.3^{11}}-\frac{4.9.14}{5.10.15.20.3^{15}}-\&c.$$

Ex. 8. Here $P=a$, $Q=\frac{\pm x}{a}$, and $\frac{m}{n}=\frac{1}{2}$,

Therefore

$$P^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{\frac{1}{2}}=A,$$

$$\frac{m}{n}AQ=\frac{1}{2}\times\frac{a^{\frac{1}{2}}}{1}\times\frac{\pm x}{a}=\frac{\pm x}{2a}a^{\frac{1}{2}}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{\pm xa^{\frac{1}{2}}}{2a}\times\frac{\pm x}{a}=\frac{-x^2}{2.4a^2}a^{\frac{1}{2}}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times\frac{-x^2}{2.4a^2}a^{\frac{1}{2}}\times\frac{\pm x}{a}=\frac{\pm 3x^3}{2.4.6a^3}a^{\frac{1}{2}}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{\pm 3x^3}{2.4.6a^3}a^{\frac{1}{2}}\times\frac{\pm x}{a}=\frac{-3.5x^4}{2.4.6.8a^4}a^{\frac{1}{2}}=E,$$

Whence $(a\pm x)^{\frac{1}{2}}=$

$$a^{\frac{1}{2}}\left\{1\pm\frac{x}{2a}-\frac{x^2}{2.4a^2}\pm\frac{3x^3}{2.4.6.a^3}-\frac{3.5x^4}{2.4.6.8a^4}\pm\&c.\right\}$$

Ex. 9. Here we have $P=a$, $Q=\frac{\pm b}{a}$, and $\frac{m}{n}=\frac{1}{3}$,

Whence

$$P^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{\frac{1}{3}}=A,$$

$$\frac{m}{n}AQ=\frac{1}{3}\times\frac{a^{\frac{1}{3}}}{1}\times\frac{\pm b}{a}=\frac{\pm b}{3a}a^{\frac{1}{3}}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{\pm b}{3a}a^{\frac{1}{3}}\times\frac{\pm b}{a}=\frac{-2b^2}{3.6a^2}a^{\frac{1}{3}}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times\frac{-2b^2}{3.6a^2}a^{\frac{1}{3}}\times\frac{\pm b}{a}=\frac{\pm 2.5b^3}{3.6.9a^3}a^{\frac{1}{3}}=D,$$

From which the law of continuation is obvious; and therefore we have $(a\pm b)^{\frac{1}{3}}=$

$$a^{\frac{1}{3}}\{1\pm\frac{b}{3a}-\frac{2b^2}{3.6a^2}\pm\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}\pm \&c.\}$$

Ex. 10. Here $P=a$, $Q=\frac{-b}{a}$, and $\frac{m}{n}=\frac{1}{4}$,

Therefore

$$P^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{\frac{1}{4}}=A,$$

$$\frac{m}{n}AQ=\frac{1}{4}\times\frac{a^{\frac{1}{4}}}{1}\times\frac{-b}{a}=\frac{-b}{4a}a^{\frac{1}{4}}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-4}{8}\times\frac{-b}{4a}a^{\frac{1}{4}}\times\frac{-b}{a}=\frac{-3b^2}{4.8a^2}a^{\frac{1}{4}}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-8}{12}\times\frac{-3b^2}{4.8a^2}a^{\frac{1}{4}}\times\frac{-b}{a}=\frac{-3.7b^3}{4.8.12a^3}a^{\frac{1}{4}}=D.$$

Where again the law of the series is discovered; and we have $(a-b)^{\frac{1}{4}}=$

$$a^{\frac{1}{4}}\{1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-\&c.\}$$

In the last three examples we have repeated the fractional root of the first term in every line; but it is obvious

that this is not necessary, as we may leave it out of every term; only remembering to introduce it at last as a general multiplier of the series.

Or we might have put our examples under a different form, as in the following instance:

Ex. 11. Here the proposed quantity may be put under the form $a^{\frac{2}{3}} \times (1+\frac{x}{a})^{\frac{2}{3}}$; and therefore, omitting for the present the multiplier $a^{\frac{2}{3}}$, we have

$\text{P}=1$, $\text{Q}=\frac{x}{a}$, and $\frac{m}{n}=\frac{2}{3}$; whence

$$\text{P}^{\frac{m}{n}}=1^{\frac{m}{n}}=1^{\frac{2}{3}}=1=\text{A},$$

$$\frac{m}{n}\text{AQ}=\frac{2}{3}\times\frac{1}{1}\times\frac{x}{a}=\frac{2x}{3a}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{2-3}{6}\times\frac{2x}{3a}\times\frac{x}{a}=\frac{-2x^2}{3.6a^2}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{2-6}{9}\times\frac{-2x^2}{3.6a^2}\times\frac{x}{a}=\frac{2.4x^3}{3.6.9a^3}=\text{D},$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{2-9}{12}\times\frac{2.4x^3}{3.6.9a^3}\times\frac{x}{a}=\frac{-2.4.7x^4}{3.6.9.12a^4}=\text{E},$$

Whence $(a+x)^{\frac{2}{3}}$, or $a^{\frac{2}{3}}(1+\frac{x}{a})^{\frac{2}{3}}=$

$$a^{\frac{2}{3}}\times\{1+\frac{2x}{3a}-\frac{2x^2}{3.6a^2}+\frac{2.4x^3}{3.6.9a^3}-\frac{2.4.7x^4}{3.6.9.12a^4}+\&c.\}$$

Which, by cancelling the multiplier 2, in the numerator and denominator, becomes

$$a^{\frac{2}{3}}\times\{1+\frac{2x}{3a}-\frac{x^2}{9a^2}+\frac{4x^3}{9^2a^3}-\frac{4.7x^4}{9^2.12a^4}+\&c.\}$$

Ex. 12. Here $\text{P}=1$, $\text{Q}=-x$, and $\frac{m}{n}=\frac{2}{5}$; whence

we have $\text{P}^{\frac{m}{n}}=1^{\frac{m}{n}}=1^{\frac{2}{5}}=1=\text{A},$

$$\frac{m}{n}\text{AQ}=\frac{2}{5}\times\frac{1}{1}\times\frac{-x}{1}=\frac{-2x}{5}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{2-5}{10}\times\frac{-2x}{5}\times\frac{-x}{1}=\frac{-2.3x^2}{5.10}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{2-10}{15}\times\frac{-2.3x^2}{5.10}\times\frac{-x}{1}=\frac{-2.3.8x^3}{5.10.15}=\text{D}$$

Therefore $(1-x)^{\frac{2}{5}}=$

$$1-\frac{2x}{5}-\frac{2.3x^2}{5.10}-\frac{2.3.8x^3}{5.10.15}-\frac{2.3.8.13x^4}{5.10.15.20}-\&c.$$

Ex. 13. Here $\dfrac{1}{(a\pm x)^{\frac{1}{2}}}=(a\pm x)^{-\frac{1}{2}}$; therefore

$$\text{P}=a,\ \text{Q}=\frac{\pm x}{a},\ \text{and}\ \frac{m}{n}=\frac{-1}{2}$$

Whence we have $\text{P}^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{-\frac{1}{2}}=\text{A}$,

And, by omitting this factor in the subsequent operations,

$$\frac{m}{n}\text{AQ}=\frac{-1}{2}\times\frac{\pm x}{a}=\frac{\mp x}{2a}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-1-2}{4}\times\frac{\mp x}{2a}\times\frac{\pm x}{a}=\frac{+3x^2}{2.4a^2}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{-1-4}{6}\times\frac{+3x^2}{2.4a^2}\times\frac{\pm x}{a}=\frac{\mp 3.5x^3}{2.4.6a^3}=\text{D},$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{-1-6}{8}\times\frac{\mp 3.5x^3}{2.4.6a^3}\times\frac{\pm x}{a}=\frac{+3.5.7x^4}{2.4.6.8a^4}=\text{E}.$$

Whence, introducing the general factor $a^{-\frac{1}{2}}$, or $\dfrac{1}{a^{\frac{1}{2}}}$,

we have $\dfrac{1}{(a\pm x)^{\frac{1}{2}}}=$

$$\frac{1}{a^{\frac{1}{2}}}\left\{1\mp\frac{x}{2a}+\frac{3x^2}{2.4a^2}\mp\frac{3.5x^3}{2.4.6a^3}+\frac{3.5.7x^4}{2.4.6.8a^4}\mp\&c.\right\}$$

Ex. 14. Here $\dfrac{a}{(a\pm x)^{\frac{1}{3}}}=a(a\pm x)^{-\frac{1}{3}}$; therefore

$$\text{P}=a,\ \text{Q}=\frac{\pm x}{a},\ \text{and}\ \frac{m}{n}=\frac{-1}{3}.$$

Whence

$$P^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{-\frac{1}{3}}=\frac{1}{a^{\frac{1}{3}}},=A,\text{ (and omitting this term)}$$

$$\frac{m}{n}AQ=\frac{-1}{3}\times\frac{\pm x}{a}=\frac{\mp x}{3a}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-1-3}{6}\times\frac{\mp x}{3a}\times\frac{\pm x}{a}=\frac{+4x^2}{3.6a^2}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-1-6}{9}\times\frac{4x^2}{3.6a^2}\times\frac{\pm x}{a}=\frac{\mp 4.7x^3}{3.6.9a^3}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-1-9}{12}\times\frac{\mp 4.7x^3}{3.6.9a^3}\times\frac{\pm x}{a}=\frac{+4.7.10x^4}{3.6.9.12a^4}=E,$$

Whence, introducing our two general factors $a\times\frac{1}{a^{\frac{1}{3}}}=a^{\frac{2}{3}}$, we have $a(a\pm x)^{-\frac{1}{3}}=$

$$a^{\frac{2}{3}}\{1\mp\frac{x}{3a}+\frac{4x^2}{3.6a^2}\mp\frac{4.7x^3}{3.6.9a^3}+\frac{4.7.10x^4}{3.6.9.12a^4}\mp\text{\&c.}\}$$

Ex. 15. Here $\frac{1}{(1+x)^{\frac{1}{5}}}=(1+x)^{-\frac{1}{5}}$; consequently

$$P=1,\ Q=\frac{x}{1},\text{ and }\frac{m}{n}=\frac{-1}{5},$$

Whence, in this case,

$$P^{\frac{m}{n}}=1^{\frac{m}{n}}=1^{-\frac{1}{5}}=1=A,$$

$$\frac{m}{n}AQ=\frac{-1}{5}\times\frac{1}{1}\times\frac{x}{1}=\frac{-x}{5}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-1-5}{10}\times\frac{-x}{5}\times\frac{x}{1}=\frac{+6x^2}{5.10}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-1-10}{15}\times\frac{6x^2}{5.10}\times\frac{x}{1}=\frac{-6.11x^3}{5.10.15}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-1-15}{20}\times\frac{-6.11x^3}{5.10.15}\times\frac{x}{1}=\frac{+6.11.16x^4}{5.10.15.20}=E,$$

Therefore $\dfrac{1}{(1+x)^{\frac{1}{5}}}=$

$$1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-\&c.$$

Ex. 16. Here $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}=\dfrac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}=\dfrac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}\times$ $\dfrac{(a+x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}}=\dfrac{a+x}{(a^2-x^2)^{\frac{1}{2}}}=(a+x)\ (a^2-x^2)^{-\frac{1}{2}}$. Whence, omitting, till the expansion is effected, the leading factor, $a+x$; we have $\text{P}=a^2$, $\text{Q}=\dfrac{-x^2}{a^2}$ and $\dfrac{m}{n}=\dfrac{-1}{2}$.

And consequently

$$\text{P}^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{-\frac{1}{2}}=a^{-1}=\frac{1}{a}=\text{A},$$

$$\frac{m}{n}\text{AQ}=\frac{-1}{2}\times\frac{1}{a}\times\frac{-x^2}{a^2}=\frac{x^2}{2a^3}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-1-2}{4}\times\frac{x^2}{2a^3}\times\frac{-x^2}{a^2}=\frac{3x^4}{2.4a^5}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{-1-4}{6}\times\frac{3x^4}{2.4a^5}\times\frac{-x^2}{a^2}=\frac{3.5x^6}{2.4.6a^7}=\text{D},$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{-1-6}{8}\times\frac{3.5x^6}{2.4.6a^7}\times\frac{-x^2}{a^2}=\frac{3.5.7x^8}{2.4.6.8a^9}=\text{E},$$

Or $(a^2-x^2)^{-\frac{1}{2}}=$

$$\frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{2.4a^5}+\frac{3.5x^6}{2.4.6a^7}+\&c.$$

Mult. by $a+x$

$$1+\frac{x^2}{2a^2}+\frac{3x^4}{2.4a^4}+\frac{3.5.x^6}{2.4.6a^6}+\&c.$$

$$+\frac{x}{a}+\frac{x^3}{2a^3}+\frac{3x^5}{2.4a^5}+\frac{3.5x^7}{2.4.6a^7}+\&c.$$

Ans. $1+\dfrac{x}{a}+\dfrac{x^2}{2a^2}+\dfrac{x^3}{2a^3}+\dfrac{3x^4}{2.4a^4}+\dfrac{3x^5}{2.4a^5}+\&c.$

Or, the expression in this example may be reduced to a more simple form by taking

$$\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}=\left(1-\frac{2x}{a+x}\right)$$

But in order to have the proper answer in this case, the terms of the result must be divided by $a+x$ and its powers.

Which being done, we have, as above,

$$1+\frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{2a^3}+\frac{3x^4}{2.4a^4}+\frac{3x^5}{2.4a^5}+\&c. \text{ Ans.}$$

INDETERMINATE ANALYSIS.

PROBLEM I.

EXAMPLES FOR PRACTICE.

Ex. 1. Given equation $3x=8y-16$.

Here $x=\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}=wh$; and

$$\therefore \frac{2y-1}{3}=wh, \text{ or } \frac{3y}{3}-\frac{2y-1}{3}=\frac{y+1}{3}=wh=p.$$

Whence $y=3p-1$,

And $x=\frac{8y-16}{3}=8p-8$, in which p can neither be 0 nor 1; taking $\therefore p=2$, we have $x=8$, and $y=5$, for the least values.

Ex. 2. Given equation $14x=5y+7$.

Here it may be observed, that since $14x$ and 7 are both divisible by 7, $5y$ must be so likewise; consequently y must be divisible by 7. Let therefore $y=7z$, and we have $5y=35z$; which substituted for $5y$, gives the equation

$$14x=35z+7, \text{ or } 2x=5z+1;$$

Whence $x=\frac{5z+1}{2}=2z+\frac{z+1}{2}=wh$.

Therefore $\frac{z+1}{2}=p$, or $z=2p-1$; whence $x=\frac{5z+1}{2}=\frac{10p-4}{2}=5p-2$; and $y=7z=14p-7$.

Hence the general values of x and y, are $x=5p-2$, and $y=14p-7$, where p may be assumed $=1$, or any integer whatever; $p=1$, gives $x=3$, and $y=7$ the least values.

Ex. 3. Given equation $27x=1600-16y$.

Here again it will be observed, that since both the terms on the right-hand side are divisible by 16, the left-hand member, viz. $27x$, must be so likewise; which cannot be except x itself be divisible by 16; make therefore $x=16z$, and our equation becomes

$$27\cdot16z=1600-16y, \text{ or}$$
$$27z=100-y;$$

Whence $y=100-27z$; and $x=16z$, where z may be assumed at pleasure, provided $27z$ be less than 100.

If $z=1$, then $x=16$, and $y=73$,
$z=2$, then $x=32$, and $y=46$,
$z=3$, then $x=48$, and $y=19$,

Which are the only four answers to the question.

Ex. 4. Let $7x$ and $11y$ be the two parts required, then we have $7x+11y=100$.

Whence $x=\dfrac{100-11y}{7}=14-2y+\dfrac{2+3y}{7}$.

Make now $\dfrac{2+3y}{7}=p$; and we have $3y=7p-2$, or $y=\dfrac{7p-2}{3}=2p-1+\dfrac{p+1}{3}$. Again, make $\dfrac{p+1}{3}=q$, and we obtain $p=3q-1$.

Therefore $y=\dfrac{7p-2}{3}=\dfrac{21q-7-2}{3}=7q-3$,

And $x=\dfrac{100-11y}{7}=\dfrac{100-77q+33}{7}=19-11q$,

where it is obvious that q cannot be taken greater than 1; making therefore $q=1$, we have $x=19-11q=8$, and $y=7q-3=4$; therefore $7x=56$ is one part, and $11y=44$ the other.

Ex. 5. Given equation $9x+13y=2000$.

Whence $x=\dfrac{2000-13y}{9}=222-y+\dfrac{2+4y}{9}=wh$

Make now $\frac{2-4y}{9}=p$, or $4y=2-9p$, or

$$y=\frac{2-9p}{4}=-2p+\frac{2-p}{4}=wh.$$

Then, taking $\frac{2-p}{4}=q$, we have $p=2-4q$,

Hence $y=\frac{2-9p}{4}=\frac{2-18+36q}{4}=9q-4$, and $x=\frac{2000-13y}{9}=\frac{2000-117q+52}{9}=228-13q$.

Where q may be assumed at pleasure, provided only that $13q$ be less than 228; whence it may be any integer from 1 to 17; which latter therefore denotes the number of possible solutions.

If $q=1$, then $y=9q-4=5$, and $x=228-13q=215$
If $q=2$, then $y=14$, and $x=202$
If $q=3$, then $y=23$, and $x=189$
If $q=4$, then $y=32$, and $x=176$
And so on for other values of q.

Ex. 6. Given equation $11x+5y=254$.

Here $y=\frac{254-11x}{5}=51-2x-\frac{1+x}{5}=wh$;

Whence making $\frac{1+x}{5}=p$, we have $x=5p-1$,

And $y=\frac{254-11x}{5}=\frac{254-55p+11}{5}=53-11p$.

Where p must be assumed less than $\frac{53}{11}$; that is $=$ any number from 1 to 4.

If $p=1$, then $x=5p-1=4$, and $y=53-11p=42$
$p=2$, then $x=\ \ 9$; and $y=31$
$p=3$, then $x=14$; and $y=20$
$p=4$, then $x=19$; and $y=9$.

Ex. 7. Given equation $17x+19y+21z=400$.

In questions of this kind, in which only the number of solutions is sought, the answer is more readily obtained from the following rule:

Let $ax+by=c$ be any proposed indeterminate equation, and find the value of p and q in the equation $ap-bq=1$; then the number of possible solutions of the equation $ax+by=c$, is equal to the difference between the integral parts of the fractions $\frac{cp}{b}-\frac{cq}{a}$.*

In our proposed equation, by transposing $21z$, we have $17x+19y=400-21z$; and by giving to z the several values, 1, 2, 3, 4, &c. we have the following set of equations; p in the equation $17p-19q=1$, being $=9$, and $q=8$;

Equations.	No. of Solutions.
$17x+19y=379$;	$\frac{9.379}{19}-\frac{8.379}{17}=1$
$17x+19y=358$;	$\frac{9.358}{19}-\frac{8.358}{17}=1$
$17x+19y=337$;	$\frac{9.337}{19}-\frac{8.337}{17}=1$
$17x+19y=316$,	$\frac{9.316}{19}-\frac{8.316}{17}=1$
$17x+19y=295$;	$\frac{9.295}{19}-\frac{8.295}{17}=1$
$17x+19y=274$;	$\frac{9.274}{19}-\frac{8.274}{17}=1$
$17x+19y=253$;	$\frac{9.253}{19}-\frac{8.253}{17}=0$
$17x+19y=232$;	$\frac{9.232}{19}-\frac{8.232}{17}=0$
$17x+19y=211$ ·	$\frac{9.211}{19}-\frac{8.211}{17}=0$
$17x+19y=190$;	†$\frac{9.190}{19}-\frac{8.190}{17}=0$

* See for this, and other questions of a similar kind, my *Treatise on Algebra*, Vols. I. and II , 2nd Edit. 1820, Octavo.

† When any of the left-hand fractions are exactly equal to an integer, the quotient must be diminished by a unit.

$$17x+19y=169\,; \qquad \frac{9.169}{19}-\frac{8.169}{17}=1$$

$$17x+19y=148\,; \qquad \frac{9.148}{19}-\frac{8.148}{17}=1$$

$$17x+19y=127\,; \qquad \frac{9.127}{19}-\frac{8.127}{17}=1$$

$$17x+19y=106\,; \qquad \frac{9.106}{19}-\frac{8.106}{17}=1$$

$$17x+19y=\ 85\,; \qquad \frac{9.85}{19}-\frac{8.85}{17}=0$$

$$17x+19y=\ 64\,; \qquad \frac{9.64}{19}-\frac{8.64}{17}=0$$

$$17x+19y=\ 43\,; \qquad \frac{9.43}{19}-\frac{8.43}{17}=0$$

$$17x+19y=\ 22\,; \qquad \frac{9.22}{19}-\frac{8.22}{17}=0$$

Total number of solutions $=$ 10 Ans.

Ex. 8. Given equation $5x+7y+11z=224$.

Here z may have any value from 1 to 19, which gives the following sets of equations; also in the equation $5p-7q=1$; we have $p=3$ and $q=2$. Hence

Equations.	No. of Solutions.
$5x+7y=213$;	$\frac{3.213}{7}-\frac{2.213}{5}=6$
$5x+7y=202$;	$\frac{3.202}{7}-\frac{2.202}{5}=6$
$5x+7y=191$;	$\frac{3.191}{7}-\frac{2.191}{5}=5$
$5x+7y=180$;	$\frac{3.180}{7}-\frac{2.180}{5}=5$
$5x+7y=169$;	$\frac{3.169}{7}-\frac{2.169}{5}=5$
$5x+7y=158$;	$\frac{3.158}{7}-\frac{2.158}{5}=4$

$$5x+7y=147;\qquad \frac{3.147}{7}-\frac{2.147}{5}=4$$

$$5x+7y=136;\qquad \frac{3.136}{7}-\frac{2.136}{5}=4$$

$$5x+7y=125;\qquad \frac{3.125}{7}-\frac{2.125}{5}=3$$

$$5x+7y=114;\qquad \frac{3.114}{7}-\frac{2.114}{5}=3$$

$$5x+7y=103;\qquad \frac{3.103}{7}-\frac{2.103}{5}=3$$

$$5x+7y=\ 92;\qquad \frac{3.92}{7}-\frac{2.92}{5}=3$$

$$5x+7y=\ 81;\qquad \frac{3.81}{7}-\frac{2.81}{5}=2$$

$$5x+7y=\ 70;\qquad \frac{3.70}{7}-\frac{2.70}{5}=1$$

$$5x+7y=\ 59;\qquad \frac{3.59}{7}-\frac{2.59}{5}=2$$

$$5x+7y=\ 48;\qquad \frac{3.48}{7}-\frac{2.48}{5}=1$$

$$5x+7y=\ 37;\qquad \frac{3.37}{7}-\frac{2.37}{5}=1$$

$$5x+7y=\ 26;\qquad \frac{3.26}{7}-\frac{2.26}{5}=1$$

$$5x+7y=\ 15;\qquad \frac{3.15}{7}-\frac{2.15}{5}=0$$

Total number of solutions $=$ 59

Ex. 9. Let x be the number of half-guineas, and y the number of half-crowns; then the number of sixpences in each of the former being 21, and in each of the latter 5: also the whole number of sixpences being 800, we have the following equation.

$$21x+5y=800.$$

Here the equation $21p-5q=1$, gives $p=1$, and $q=4$.

Whence by the rule $\frac{1.800}{5}-\frac{4.800}{21}=7$. Ans.

Where, as in the preceding examples, the first quotient is diminished by unity, being integral.

This rule, which is taken from my Treatise on Algebra, 2 vols. 8vo., Edit. 1820, is much shorter than that which depends upon an actual determination of the several solutions; but the latter is omitted here as presenting no difficulty to the student who has attended to the preceding solutions.

Ex. 10. Let x represent the number of guineas I have to give, and y the number of louis-d'ors I am to receive; then by the question,

$$21x-17y=1.$$

Hence $17y=21x-1$, or $y=x+\frac{4x-1}{17}=wh.$

Make $\frac{4x-1}{17}=p$; then $4x=17p+1$, or

$$x=4p+\frac{p+1}{4}=wh.$$

Where, if we make $\frac{p+1}{4}=q$; then $p=4q-1$,

And consequently $x=\frac{17p+1}{4}=17q-4$,

Whence also we have $y=\frac{21x-1}{17}=21q-5$;

Where q may be taken any number at pleasure: if we take $q=1$; then $x=13$, and $y=16$, which are the least numbers; viz., I must give 13 guineas, and receive 16 louis.

Ex. 11. Let x, y, and z, be the number of gallons of each sort respectively; then by the question

$$x+\quad y+\quad z=1000$$
$$12x+15y+18z=17(x+y+z)$$

By transposing the terms of the latter equation, we have

$$\begin{array}{ll} & 5x+2y-z=0 \\ \text{But} & x+y+z=1000 \\ \hline \text{By addition} & 6x+3y \qquad =1000 \end{array}$$

Where it is obvious the answer cannot be obtained in integers, because the first side of the equation is divisible by 3, and the other is not. We may therefore assume x or y at pleasure; taking for x the value given in the answer of the Introduction, viz. $x=111\frac{1}{9}$, we have

$$x=111\tfrac{1}{9};\ y=\frac{1000-6(111\frac{1}{9})}{3}=\frac{333\frac{1}{3}}{3}=111\tfrac{1}{9}$$

$$\text{and } z=5x+2y=777\tfrac{1}{9};$$

That is $111\frac{1}{9}$ at 12*s.*; $111\frac{1}{9}$ at 15*s.*; and $777\frac{7}{9}$ at 18*s.*

PROBLEM II.

Ex. 3. Let $x =$ the number sought; then by the question

$$\frac{x-2}{6}, \text{ and } \frac{x-3}{13}= \text{whole numbers.}$$

$$\text{Make } \frac{x-2}{6}=p;\ \text{then } x=6p+2.$$

Substitute this value of x in the second equation, and we have $\dfrac{6p+2-3}{13}=\dfrac{6p-1}{13}=wh.$

Make $\dfrac{6p-1}{13}=q$; then $p=\dfrac{13q+1}{6}=2q+\dfrac{q+1}{6}=wh.$

Let now $\dfrac{q+1}{6}=r$; then $q=6r-1$; and $p=$ $\dfrac{13(6r-1)+1}{6}=13r-2$; and consequently $x=6(13r-2)$ $+2=78r-10$; where r may be taken at pleasure.

If $r=1$, then $x=68$.

Ex. 4. Let x be the number sought; then by the question $\dfrac{x-5}{7}$ and $\dfrac{x-2}{9}=$ whole numbers.

$$\text{Make } \frac{x-5}{7}=p, \text{ or } x=7p+5.$$

Substitute this value for x in the second equation, and we have $\dfrac{7p+5-2}{9}=\dfrac{7p+3}{9}=wh.$

Make $\dfrac{7p+3}{9}=q$; and we have $p=\dfrac{9q-3}{7}$

$$=q+\frac{2q-3}{7}=wh.$$

Let now $\dfrac{2q-3}{7}=r$, and we shall have $q=\dfrac{7r+3}{2}$

$$=3r+1+\frac{r+1}{2}=wh.$$

Again, let there be taken $\dfrac{r+1}{2}=s$, or $r=2s-1$.

Then $q=\dfrac{7(2s-1)+3}{2}=7s-2$, and $p=\dfrac{9(7s-2)-3}{7}$ $=9s-3$; whence we have $x=7(9s-3)+5=63s-16$; where s may be taken at pleasure. If $s=1$, then $x=47$, and if $s=2$, $x=110$, &c.

Ex. 5. Here by the question we have

$$\frac{x-16}{39} \text{ and } \frac{x-27}{56} = \text{ whole numbers.}$$

Make $\dfrac{x-16}{39}=p$, or $x=39p+16$.

This, substituted in the second equation, gives

$$\frac{39p+16-27}{56}=\frac{39p-11}{56}=p-\frac{17p+11}{56}=wh.$$

Make $\dfrac{17p+11}{56}=q$,

then $p=\dfrac{56q-11}{17}=3q-1+\dfrac{5q+6}{17}$;

$$\therefore \frac{5q+6}{17}=wh.=r; \text{ and}$$

$$q=\frac{17r-6}{5}=3r-1+\frac{2r-1}{5}=wh.$$

Make $\dfrac{2r-1}{5}=s$; then $r=\dfrac{5s+1}{2}=2s+\dfrac{s+1}{2}=wh.$

Let therefore $\frac{s+1}{2}=t$, and we have $s=2t-1$;

Consequently $r=\frac{5(2t-1)+1}{2}=5t-2$, and

$q=\frac{17(5t-2)-6}{5}=17t-8$; $p=\frac{56(17t-8)-11}{17}$

$=56t-27$; and $x=39(56t-27)+16=2184t-1037$;
where t may be assumed at pleasure.

When $t=1$, then $x=1147$, the least number agreeing with the conditions.

Ex. 6. Here putting x for the number sought, we must have

$$\frac{x-5}{7}, \frac{x-7}{8}, \text{ and } \frac{x-8}{9}, \text{ all whole numbers.}$$

From the first, by putting $\frac{x-5}{7}=p$, we obtain $x=7p+5$;

and this substituted in the other two, gives

$$\frac{7p-2}{8} \text{ and } \frac{7p-3}{9} = \text{ whole numbers;}$$

Make $\frac{7p-2}{8}=q$; then $p=\frac{8q+2}{7}=q+\frac{q+2}{7}=wh.$

Let now $\frac{q+2}{7}=r$, and we have $q=7r-2$;

Whence $p=\frac{8(7r-2)+2}{7}=8r-2.$

Now, substituting this value of p in $\frac{7p-3}{9}$ we shall have

$\frac{7(8r-2)-3}{9}=\frac{56r-17}{9}=$ a whole number, or

$$6r-2+\frac{2r+1}{9}=wh.$$

Let therefore $\frac{2r+1}{9}=s$, or $r=\frac{9s-1}{2}=4s+\frac{s-1}{2}$

Where $\frac{s-1}{2}$ is likewise $=wh.=t$, and $s=2t+1$.

Then, from this we readily have the following values:

$$s=2t+1\ ;\ r=\frac{9(2t+1)-1}{2}=9t+4,$$

$$q=7(9t+4)-2=63t+28-2=63t+26,$$

$$p=\frac{8(63t+26)+2}{7}=72t+30, \text{ and}$$

$$x=7(72t+30)+5=504t+215,$$

where t may be assumed at pleasure.

If $t=0$, then $x=215$;
If $t=1$, then $x=\ 719$; if $t=2$, then $x=1223$;
If $t=3$, then $x=1727$, and so on.

Ex. 7. Let x be the number sought, then by the question

$\frac{x}{9}, \frac{x}{8}, \frac{x}{7}, \frac{x}{6}, \frac{x}{5}, \frac{x}{4}, \frac{x}{3}$, and $\frac{x}{2}$, must be all whole numbers.

Now first, if $\frac{x}{9}$ and $\frac{x}{8}$ be whole numbers, $\frac{x}{6}, \frac{x}{4}, \frac{x}{3}$, and $\frac{x}{2}$, must necessarily be so. We have, therefore, only to find $\frac{x}{9}, \frac{x}{8}, \frac{x}{7}, \frac{x}{5}$, whole numbers; which must have place if x be made equal to 2520, the product of all these denominators.

Ex. 8. Here, if we put x for the number, the conditions are that

$$\frac{x}{2}, \frac{x}{3}, \frac{x}{4}, \frac{x}{5}, \frac{x}{6}, \text{ and } \frac{x-5}{7}$$

must be all integers, or whole numbers.

But the first five of these fractions, when brought to a common denominator, are

$$\frac{30x}{60}, \frac{20x}{60}, \frac{15x}{60}, \frac{12x}{60}, \text{ and } \frac{10x}{60}\ ;$$

Whence, as any multiple of a whole number is a whole number, we have only to make $\frac{x}{60}$ and $\frac{x-5}{7}$ whole numbers.

Or, as in the preceding examples, these conditions may be reduced, by observing that if $\frac{x}{2}, \frac{x}{5}$, and $\frac{x}{6}$, are whole

numbers, $\frac{x}{3}$ and $\frac{x}{4}$ must be so likewise;

But the least value of x, which answers the three former conditions, is $x=2\times5\times6=60$; x therefore must be some multiple of 60.

Let then $x=60p$, and the last condition is that

$$\frac{60p-5}{7}=wh.$$

Now $\frac{60p-5}{7}=9p-1-\frac{3p-2}{7}$; therefore $\frac{3p-2}{7}=wh.$

Make $\frac{3p-2}{7}=q$, then $p=\frac{7q+2}{3}=2q+1+\frac{q-1}{3}$.

Again, let $\frac{q-1}{3}=r$, and we have $q=3r+1$, and consequently $p=\frac{7(3r+1)+2}{3}=7r+3$.

Where r may be assumed $=0$, or any integer whatever. If $r=0$, then $p=3$, and $x=60p=180$.

DIOPHANTINE ANALYSIS.

Ex. 1. Since $x+1$, and $x-1$, are to be both squares, let $x=y^2-1$, then $x+1=y^2$, which fulfils the first condition; and therefore it only remains to make $x-1=\square$,

$$\text{or } y^2-2=\square.$$

Let $y^2-2=(y-1)^2=y^2-2y+1$, and we shall have

$$2y=3, \text{ or } y=\tfrac{3}{2}, \text{ or } y^2=\tfrac{9}{4};$$

Consequently $x=y^2-1=\frac{9}{4}-1=\frac{5}{4}$, the number sought.

Ex. 2. Here we have to make $x+128$, and $x+192$, both squares.

First, let $x=y^2-128$, so shall $x+128=y^2$, which is the first condition;

And therefore it only remains to make $x+192=y^2+64$ a square.

Let $y^2+64=(y+r)^2=y^2+2ry+r^2$, then $64=2ry+r^2$; and $y=\frac{64-r^2}{2r}$; where r may be taken any number less than 8. If $r=2$, then $y=15$, and $x=y^2-128=97$, the number sought.

Ex. 3. Here the conditions are, that x^2+x, and x^2-x, shall be both squares.

Let $x^2+x=(r-x)^2=r^2-2rx+x^2$; then $x=\frac{r^2}{2r+1}$, which value of x, substituted in the second equation, gives $\left(\frac{r^2}{2r+1}\right)^2-\frac{r^2}{2r+1}$, or $\frac{r^2}{(2r+1)^2}\times(r^2-2r-1)$ a square;

And since the first factor is a square, we have only to find r^2-2r-1 a square.

Assume $r^2-2r-1=(r-s)^2=r^2-2rs+s^2$; then $2rs-2r=s^2+1$, or $r=\frac{s^2+1}{2(s-1)}$; and $x=\frac{r^2}{2r+1}=\frac{(s^2+1)^2}{4s(s^2-1)}$,

Where s may be any number greater than 1.

If we take $s=2$; then $x=\frac{25}{24}$ as required.

Ex. 4. Here, if we call x and y the two numbers, we have to find

$$x+xy=\square$$
$$y+xy=\square.$$

Now if we assume $x=m^2$ and $y=n^2$, these become

$$m^2+m^2n^2=m^2(1+n^2)$$
$$n^2+n^2m^2=n^2(1+m^2)$$

which must be both squares; that is, we must have $1+n^2$ and $1+m^2$ both squares.

Assume $n^2+1=(n+r)^2=n^2+2rn+r^2$;

Then, from this we have $n=\frac{1-r^2}{2r}$; and in the same manner, by putting $m^2+1=(m+s)^2$, we have $m=\frac{1-s^2}{2s}$;

Consequently $x=\left(\frac{1-s^2}{2s}\right)^2$ and $y=\left(\frac{1-r^2}{2r}\right)^2$, where s and r may be assumed at pleasure;

If $s=3$, and $r=2$; then $x=\frac{16}{9}$ and $y=\frac{9}{16}$.

Which numbers answer the conditions of the question; but they are both square numbers, which is not a necessary condition.

The question may be otherwise resolved as follows:
Let x and $x-1$ be the two numbers: then

$$x+x(x-1)=x^2 \text{ a square number.}$$

Also $(x-1)+x(x-1)=x^2-1$ is to be made a square.
Make $x^2-1=(x-r)^2=x^2-2rx+r^2$,

And we have $x=\frac{r^2+1}{2r}$; where r may be any number greater than 1.

If $r=3$, then $x=\frac{5}{3}$ and $x-1=\frac{2}{3}$, which numbers answer the conditions of the question.

Ex. 5. Let x^2, y^2, and z^2, represent the required squares; hence the condition

$$x^2+z^2=2y^2.$$

Let $x=m+n$, and $z=m-n$, then

$$x^2+z^2=2m^2+2n^2$$

$$\text{or } m^2+n^2=y^2.$$

Let $m=p^2-q^2$ and $n=2pq$, and we have

$$m^2+n^2=(p^2-q^2)^2+4pq^2=(p^2+q^2)^2=y^2.$$

Hence the following general results, viz.

$$\begin{aligned} x&=p^2-q^2+2pq, \text{ and } & x^2&=(p^2-q^2+2pq)^2 \\ y&=p^2+q^2 & y^2&=(p^2+q^2)^2 \\ z&=p^2-q^2-2pq & z^2&=(p^2-q^2-2pq)^2 \end{aligned}$$

Where p and q may be assumed at pleasure.*
If $p=2$, and $q=1$, then $x=7$, $y=5$, and $z=-1$;
Consequently the squares are 49, 25, and 1.

Ex. 6. Let $\frac{1}{2}x^2-y$, $\frac{1}{2}x^2$, and $\frac{1}{2}x^2+y$, be the three numbers in arithmetical progression;

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* Other general expressions may be deduced as follows:
The equation $x^2+z^2=2y^2$, may be put under the form $2y^2-x^2=z^2$, and this again may be represented by

$$(2y+x)^2-2(y+x)^2=z^2$$

$$\text{Make } \left.\begin{aligned} 2y+x&=p^2+2q^2 \\ y+x&=2pq \end{aligned}\right\}$$

$$\text{Whence } \begin{aligned} y&=p^2+2q^2-2pq \\ x&=4pq-p^2-2q^2 \\ z&=p^2+2q^2 \end{aligned}$$

These results are apparently different from the former, but they are readily reducible to the same, and will in their present form equally answer the required conditions.

Then we have to find x^2, x^2+y and x^2-y rational squares; or x^2+y and x^2-y = squares.

Assume $y=2rx+r^2$; then we shall have

$$x^2+y=x^2+2rx+r^2=(x+r)^2,$$

And therefore it only remains to make x^2-y,

or $x^2-2rx-r^2=\Box$.

Assume $x^2-2rx-r^2=(x-m)^2=x^2-2mx+m^2$,

Then it is obvious that $x=\dfrac{m^2+r^2}{2m-2r}$,

Where m and r may be taken at pleasure.

If $m=5$, and $r=4$; then $x=\frac{41}{2}$, and $\frac{1}{2}x^2=\frac{1681}{8}$.

Also $y=2rx+r^2=41\times4+16=180$; whence the three numbers will be $30\frac{1}{8}$, $210\frac{1}{8}$, and $390\frac{1}{8}$.

And if these numbers be multiplied by any square number, the same conditions will obviously obtain;

Hence multiplying by 4 we shall have

$120\frac{1}{2}$, $840\frac{1}{2}$, and $1560\frac{1}{2}$,

which answer the conditions of the question;

And if these last be again multiplied by 4, we shall have 482, 3362, and 6242, which are all integral; being the same as in the Introduction.

The question may also be otherwise solved, as follows:

Let x, y, and z, be the numbers, and assume

$$x+y=m^2$$
$$x+z=n^2$$
$$y+z=r^2$$

Then we shall have

$$x=\tfrac{1}{2}(\quad m^2+n^2-r^2)$$
$$y=\tfrac{1}{2}(\quad m^2-n^2+r^2)$$
$$z=\tfrac{1}{2}(-m^2+n^2+r^2)$$

Where, since the numbers are in arithmetical progression, we have $x+z=2y$; or $n^2=m^2-n^2+r^2$.

Therefore we have now only to find

$$m^2+r^2=2n^2;$$

That is three square numbers in arithmetical progression;

Whence from Example 5, we shall have

$$m^2=(p^2-q^2-2pq)^2=(p^2+q^2)^2-4pq(p^2-q^2),$$
$$n^2=(p^2+q^2)^2$$
$$r^2=(p^2-q^2+2pq)^2=(p^2+q^2)^2+4pq(p^2-q^2).$$

And consequently

$$x=\tfrac{1}{2}(p^2+q^2)^2-4pq(p^2-q^2)$$
$$y=\tfrac{1}{2}(p^2+q^2)^2$$
$$z=\tfrac{1}{2}(p^2+q^2)^2+4pq(p^2-q^2)$$

Where p and q may be any numbers taken at pleasure.

Ex. 7. Let x, y, and z, be the numbers sought; then by the question

$$x^2+y+z=m^2$$
$$y^2+x+z=q^2$$
$$z^2+x+y=r^2$$

Assume $x^2+y+z=(x+n)^2=x^2+2nx+n^2$;

$$\text{Then } x=\frac{y+z-n^2}{2n}.$$

Which value of x, substituted in the second and third equations, gives

$$y^2+\frac{y+z-n^2}{2n}+z=p^2$$

$$z^2+\frac{y+z-n^2}{2n}+y=r^2$$

Hence, assume $y^2+\dfrac{y+z-n^2}{2n}+z=(y-p)^2=y^2-2py+p^2$,

And $z^2+\dfrac{y+z-n^2}{2n}+y=(z-s)^2=z^2-2sz+s^2$.

From the 1st. $z=\dfrac{n^2-y-4pny+2np^2}{1+2n}$

From the 2d. $z=\dfrac{n^2-y-2ny+2ns^2}{1+4ns}$.

Whence

$$\frac{n^2-y-4pny+2np^2}{1+2n}=\frac{n^2-y-2ny+2ns^2}{1+4ns},$$

And consequently by reduction,

$$y=\frac{4nsp^2+2n^2s+p^2-n^2-s^2-2ns^2}{2p+2s+8nps-2-2n},$$

Where n, p, and s, may be assumed at pleasure.

If $n=1$, $p=4$, and $s=11$, then $y=1$, $z=\dfrac{16}{3}$ and $x=\dfrac{8}{3}$;

Which answer the required conditions And by taking different numbers for n, p, and s, other answers may be found.

Ex. 8. Let $8x$ and $15x$, having the required ratio, denote the two numbers; then as $(8x)^2+(15x)^2=(17x)^2$ is already a square, it is manifest that any two numbers having the ratio of 8 to 15 will answer the conditions of the question.

Ex. 9. Let x and y be the numbers sought; then by the question we have to solve the equations

$$x^2+xy, \text{ or } x(x+y)=m^2$$
$$y^2+xy, \text{ or } y(x+y)=n^2$$

Assume $x=p^2$, and $y=q^2$, then these will become

$$p^2(p^2+q^2)=m^2$$
$$q^2(p^2+q^2)=n^2$$

Which conditions will be fulfilled if we find $p^2+q^2=$ a square.

Now we have already seen (Example 5) that for this purpose we have only to assume

$$p=r^2-s^2 \text{ and } q=2rs.$$

$$\text{Therefore } x=p^2=(r^2-s^2)^2$$
$$\text{And } y=q^2=4r^2s^2$$

Where r and s may be assumed at pleasure.

If $r=2$ and $s=1$, then $x=9$ and $y=16$. These however are square numbers, which is not a necessary condition.

It is obvious, that any multiple whatever of the same numbers will equally answer the conditions of the question.

We shall have, therefore, a more general solution by taking $x=t(r^2-s^2)^2$, and $y=4tr^2s^2$; where r, s, and t, may be assumed at pleasure.

Ex. 10. Let x and y represent the two numbers; then by the question $\quad x^2+y^2-1=m^2$

$$x^2-y^2-1=n^2$$

Assume $x=y+1$; then these two equations reduce to

$$(y+1)^2+y^2-1=2y^2+2y=m^2$$
$$(y+1)^2-y^2-1= \qquad 2y=n^2$$

The first is equivalent to $2y(y+1)=m^2$, or to

$$n^2(y+1)=m^2,$$

Consequently $y+1$ must be a square; let,

therefore $\quad y+1=p^2$; then, since

$2y=n^2$, we shall have by

subtraction $\quad y=n^2-p^2+1$

And consequently $x=n^2-p^2+2$

Where n and p may be assumed at pleasure. If $n=4$, and $p=3$, then $y=8$, and $x=9$, the numbers required.

Ex. 11. Without attending to the particular numbers in the question, let us endeavour to resolve any given square number into two other square numbers.

For this purpose, let a^2 represent the given square that is to be so resolved; and put x^2 and y^2 for the required squares.

Then we have to satisfy the equation

$$a^2=x^2+y^2, \text{ or}$$
$$a^2-y^2=x^2$$

In order to which, let us assume

$$a+y=\frac{px}{q}$$

$$a-y=\frac{qx}{p}$$

From which we have, by addition and subtraction,

$$2a=\frac{px}{q}+\frac{qx}{p}=\frac{(p^2+q^2)x}{pq}$$

$$2y=\frac{px}{q}-\frac{qx}{p}=\frac{(p^2-q^2)x}{pq}$$

Whence $x=\dfrac{2pqa}{p^2+q^2}$, and $y=\dfrac{(p^2-q^2)a}{p^2+q^2}$

Where the indeterminates p and q may be assumed at pleasure.

But it is obvious that $2\ pq$ and p^2+q^2, as also p^2-q^2 and p^2+q^2, being incommensurate, the values of x and y must be fractional, except a be divisible by p^2+q^2.

That is, unless a be divisible by the sum of two squares; and then the question will admit of as many integral answers as a has divisors of this form.

Now by the question $a=65$; and 65 is divisible by 5, or 2^2+1^2; or by 13, or 3^2+2^2; and by 65, or 7^2+4^2, or 8^2+1^2.

Hence we may assume $p=2$, and $q=1$; or $p=3$, and $q=2$; or $p=7$, and $q=4$; or $p=8$, and $q=1$;

Which will give the four following solutions,

$$65^2=52^2+39^2=60^2+25^2=56^2+33^2=16^2+63^2$$

These being the only integral answers the question admits of.

Ex. 12. Let x^2, y, and $\frac{y^2}{x^2}$, be the numbers required; and put the given number $19=a$; then by the question x^2+a, $y+a$, and $\frac{y^2}{x^2}+a$, are to be all squares.

Assume $x^2+a=(r-x)^2=r^2-2rx+x^2$, and $y+a=r^2$

Then we shall have $x=\frac{r^2-a}{2r}$, and $y=r^2-a$.

Wherefore $\frac{y}{x}=2r$, and $\frac{y^2}{x^2}+a=4r^2+a$; which must be a square.

$$\text{Let } 4r^2+a=(2r+s)^2=4r^2+4rs+s^2$$

$$\text{In which case we have } r=\frac{a-s^2}{4s}$$

$$\text{And consequently } y=\left(\frac{a-s^2}{4s}\right)^2-a$$

Where s may be taken at pleasure, provided $\left(\frac{a-s^2}{4s}\right)^2$ be greater than a.

But by the question $a=19$; if, therefore, we assume $s=1$, we have $y=\left(\frac{19-1}{4}\right)^2-19=\frac{5}{4}$; $\frac{y}{x}=2r=\frac{a-s^2}{2s}=\frac{19-1}{2}=9$,

$$\text{and } x=\frac{5}{4}\div 9=\frac{5}{36}.$$

Hence the three numbers are

$$x^2=\left(\frac{5}{36}\right)^2=\frac{25}{1296},\ y=\frac{5}{4}.\ \text{and } \frac{y^2}{x^2}=^2\ =81.$$

The question may also be otherwise answered, thus:

Let $4x^2$, and x^2-a, be two of the numbers; then $\frac{(x^2-a)^2}{4x^2}$ will be the third number; and by the question

$$4x^2+a \quad =\square$$
$$x^2-a+a \quad =\square$$
$$\frac{(x^2-a)^2}{4x^2}+a=(x^2+a)^2=\square$$

Here the second and third expressions being squares, it only remains to make the first $4x^2+a=\square$.

Assume $4x^2+a=(2x+s)^2=4x^2+4sx+s^2$,

Whence $x=\frac{a-s^2}{4s}$; where s may be taken at pleasure, provided only that s^2 be less than a.

Ex. 13. Let x and y be the two numbers we have to find; then $\quad x^2+y^2+xy=\square$

Assume $x^2+xy+y^2=(x+r)^2=x^2+2rx+r^2$

And we shall have $x=\frac{y^2-r^2}{2r-y}$

Where r and y may be taken at pleasure, provided y be greater than r, but less than $2r$.

If $y=3$, and $r=2$, then $x=5$

$y=5$, and $r=3$, then $x=16$

$y=7$, and $r=6$, then $x=\frac{13}{5}$

Ex. 14. Let x, rx, and sx be the numbers; then, by the question,

$$x^2(1+rs)=\square\text{, or }1+rs=\square$$
$$x^2(r^2+s)=\square\text{, or }r^2+s=\square$$
$$x^2(s^2+r)=\square\text{, or }s^2+r=\square$$

Make $r^2+s=(r-n)^2=r^2-2nr+n^2$

and $s^2+r=(s+2)^2=s^2+4s+4$.

$\therefore\ s+2nr=n^2$, and $r-4s=4$

The resolution of the two last equations gives

$$r=\frac{4(n^2+1)}{8n+1}\text{, and }s=\frac{n(n-8)}{8n+1}$$

which values of r and s make $1+rs$ a square; hence all the conditions are satisfied, whatever be the value of x.

Take, therefore, $x=8n+1$, and we shall have $rx=4(n^2+1)$, and $sx=n(n-8)$.

Hence $n(n-8)$, $8n+1$, and $4(n^2+1)$ are general expressions for the three required numbers, in which n may be taken any number greater than 8.

Let $n=9$; then the three numbers are 9, 73, and 328.

Let $n=13$; then we shall have the set 5 (13), 5 (21), and 5 (136) answering the proposed conditions. Or by rejecting the common factor 5, we have 13, 21, and 136, which are the least positive integral numbers that the question admits of.

Ex. 15. Find three squares a^2, b^2, c^2, such that their sum may be a square, or

$$a^2+b^2+c^2=d^2.$$

That is, assume $a^2+b^2+c^2=(c+r)^2$; from which we shall obtain $c=\frac{a^2+b^2-r^2}{2r}$,

where a, b, and r, may be taken at pleasure, provided r^3 be less than a^2+b^2.

This being done, let ax, bx, and cx, be the required squares; then $a^2x^2+b^2x^2+c^2x^2=d^2x^2$

And we have to find

$$d^2x^2+ax=\square$$
$$d^2x^2+bx=\square$$
$$d^2x^2+cx=\square$$

Or dividing by d^2, and putting $\frac{a}{d^2}=m$, $\frac{b}{d^2}=n$, and $\frac{c}{d^2}=p$, we have

$$x^2+mx=\square$$
$$x^2+nx=\square$$
$$x^2+px=\square$$

Assume $x^2+mx=(r-x)^2=r^2-2rx+x^2$

Then we have $x=\frac{r^2}{2r+m}$.

Which value of x, substituted in the second and third, gives

$$\frac{r^2}{(2r+m)^2}\times\{r^2+n(2r+m)\}=\square$$
$$\frac{r^2}{(2r+m)^2}\times\{r^2+p(2r+m)\}=\square;$$

And since the first two factors of each are squares, we have only to make

$$r^2+n(2r+m)=\square$$
$$r^2+p(2r+m)=\square$$

Make $r^2+n(2r+m)=(s-r)^2=s^2-2sr+r^2$

Then we have $r=\dfrac{s^2-mn}{2(n+s)}$,

Which value of r, substituted in the latter expression, gives

$$\left(\frac{s^2-mn}{2(n+s)}\right)^2+p\left\{\frac{s^2-mn}{n+s}+m\right\}=\Box,$$

Which, by reduction, becomes

$$\frac{(s^2-mn)^2+4ps(s+m)(s+n)}{4(n+s)^2}=\Box.$$

Or $(s^2-mn)^2+4ps(s+m)(s+n)=\Box$.

Assume this $=\{(s^2-mn)-2ps\}^2=$

$(s^2-mn)^2-4ps(s^2-mn)+4p^2s^2$

Then this, by reduction, gives

$(s+m)(s+n)=mn-s^2+ps$

Whence $s=\dfrac{p-(m+n)}{2}$,

$$\text{And } x=\frac{r^2}{2r+m}=\frac{(s^2-mn)^2(n+s)}{4(s^2+ms)(n+s)^2}=\frac{(s^2-mn)^2}{4s(s+m)(s+n)}$$

Where, if we take $a=2$, $b=6$, and $c=9$, then $d=11$,

$m=\dfrac{a}{d^2}=\dfrac{2}{121}$, $n=\dfrac{b}{d^2}=\dfrac{6}{121}$, and $p=\dfrac{c}{d^2}=\dfrac{9}{121}$,

$s=\dfrac{p-(m+n)}{2}=\dfrac{9-8}{121}\div 2=\dfrac{1}{242}$, and $x=\dfrac{2209}{62920}$

Whence $ax=2x=\dfrac{4418}{62920}$

$bx=6x=\dfrac{13254}{62920}$

$cx=9x=\dfrac{19881}{62920}$ the numbers required.

Ex. 16. Let x, rx, and r^2x be the three numbers in geometrical progression; then by the question

$x+rx=x(1+r)=\Box$

$rx+r^2x=rx(1+r)=\Box$

Dividing the second by the first, it is obvious that r must be a square; let, therefore, $r=y^2$, and it only remains to find $x(1+y^2)=\square$

Which will be the case, if we make

$$x=y^2+1$$

in which y may be taken at pleasure.

If $y=2$, then $x=5$, and $r=4$; and the three numbers sought are 5, 20, and 80.

Ex. 17. Let x and y represent the two numbers; then it is required to find

$$x+1=m^2$$
$$y+1=n^2$$
$$x+y+1=r^2$$
$$x-y+1=s^2$$

Now it is obvious that the three squares r^2, m^2, and s^2, are in arithmetical progression, their common difference being y.

Let us, therefore, represent these squares as in the note to Ex. 5, page 113; viz.

$$s^2=(4pq-p^2-2q^2)^2$$
$$m^2=(p^2+2q^2-2pq)^2$$
$$r^2=(p^2-2q^2)^2$$

Then we have for their common difference

$$y=4p^3q-12p^2q^2+8pq^3$$

and all that is required is to find this quantity, $+1$, a square, or

$$4p^3q-12p^2q^2+8pq^3+1=n^2$$

Assume, therefore, in this case,

$$n=1+4pq^3$$

And we shall have, by squaring and cancelling the like parts,

$$4p^3q-12p^2q^2=16p^2q^6,$$

Whence

$$p=4q^5+3q,$$

in which expression q may be assumed at pleasure.

Thus the general values of x and y will be determined; viz. by putting $p=4q^5+3q$, and then

$$x=(p^2+2q^2-2pq)^2-1$$
$$y=(1+4pq^3)^2-1$$

Where, by taking $q=1$, we have $p=7$; whence $x=1368$ and $y=840$; which numbers answer the conditions of the question; for

$$\begin{aligned} 1368+\quad 1&=37^2\\ 840+\quad 1&=29^2\\ 1368+840+1&=47^2\\ 1368-840+1&=23^2 \end{aligned}$$

These numbers are the same as the answer in the introduction, and are the least integral numbers the question admits of; various others, however, may be found by giving different values to q.

Ex. 18. Let x, y, and z, be the three numbers, and s their sum; then we have to find

$$\begin{gathered} x^2+s=\square,\ x^2-s=\square \\ y^2+s=\square,\ y^2-s=\square \\ z^2+s=\square,\ z^2-s=\square \\ \text{and } x+y+z=s. \end{gathered}$$

Now, if we assume for r and s any numbers whatever, and take $a=r^2-s^2$, $b=2rs+s^2$, and $c=r^2+rs+s^2$, we shall have $c^2=a^2+ab+b^2$; where a, b, and c, will be known numbers, and will possess the following properties, viz.

$$\begin{aligned} &(c^2+b^2)^2 &&\pm 4abc(a+b), \text{ a complete sq.}\\ &(c^2+a^2)^2 &&\pm 4abc(a+b), \text{ a complete sq.}\\ &\{c^2+(a+b)^2\}^2 &&\pm 4abc(a+b), \text{ a complete sq.} \end{aligned}$$

Let the three last expressions be multiplied by x^2, and we shall have

$$\begin{aligned} &x^2(c^2+b^2)^2 &&\pm 4abc(a+b)x^2=\square\\ &x^2(c^2+a^2)^2 &&\pm 4abc(a+b)x^2=\square\\ &x^2\{c^2+(a+b)^2\} &&\pm 4abc(a+b)x^2=\square \end{aligned}$$

Hence all the conditions will be satisfied, if we make the sum of the square roots of the former parts of these expressions $=4abc(a+b)x^2$: viz.

$$x(c^2+b^2)+x(c^2+a^2)+x\{c^2+(a+b)^2\}= \\ 4abc(a+b)x^2,$$

$$\text{Whence } x=\frac{3c^2+2a^2+2b^2+2ab}{4abc(a+b)}$$

Or putting $c^2+b^2=m$, $c^2+a^2=n$, $c^2+(a+b)^2=p$, and $4abc(a+b)=q$

we shall have $x=\dfrac{m+n+p}{q}$

$$\text{And } mx=\frac{m}{q}(m+n+p)$$

$$nx=\frac{n}{q}(m+n+p)$$

$$px=\frac{p}{q}(m+n+p)$$

Which are the three numbers sought.

If $r=2$, and $s=1$; then $a=3$, $b=5$, and $c=7$. Also $m=c^2+b^2=74$, $n=c^2+a^2=58$; $p=c^2+(a+b)^2=113$, and $q=4abc(a+b)=3360$.

$$\text{Whence } x=\frac{m+n+p}{q}=\frac{245}{3360}=\frac{7}{96}$$

$$\text{Consequently } mx=\frac{7\times 74}{96}=\frac{518}{96}$$

$$nx=\frac{7\times 58}{96}=\frac{406}{96}$$

$$px=\frac{7\times 113}{96}=\frac{791}{96}$$

are the numbers sought; and others may be found, by assuming different values for r and s.

Ex. 19. Let x^2, y^2, and z^2, be the required squares; then we have to find

$$x^4+y^4+z^4=\square.$$

$$\text{Assume } x^4+y^4+z^4=\{(x^2+y^2)-z^2\}^2=$$
$$x^4+2x^2y^2+y^4-2z^2(x^2+y^2)+z^4$$

$$\text{Then } 2x^2y^2=2z^2x^2+2z^2y^2$$

$$\text{And consequently } z^2=\frac{x^2y^2}{x^2+y^2}$$

Therefore x^2+y^2 must be a square; which it will be if we assume $x=p^2-q^2$, and $y=2pq$; for then

$$x^2+y^2=(p^2-q^2)^2+4p^2q^2=(p^2+q^2)^2$$

Hence the required squares will now be

$$x^2=(p^2-q^2)^2$$
$$y^2=4p^2q^2$$
$$z^2=\frac{4p^2q^2(p^2-q^2)^2}{(p^2+q^2)^2}$$

Where p and q may be any numbers taken at pleasure.

If $p=2$, and $q=1$, then

$$x^2=(p^2-q^2)^2=3^2=9$$
$$y^2=4p^2q^2=4^2=16$$
$$z^2=\frac{4p^2q^2(p^2-q^2)^2}{(p^2+q^2)^2}=\frac{12^2}{5^2}=\frac{144}{25}$$

which numbers answer the required conditions; and various others may be found by giving different values to p and q.

Ex. 20. Here, if x^2, y^2, and z^2, be taken to represent the three squares, we shall have to make

$$x^2-y^2=\square$$
$$x^2-z^2=\square$$
$$z^2-y^2=\square$$

Put $x^2-y^2=p^2$, and $z^2-y^2=q^2$;

then, by subtraction, $x^2-z^2=p^2-q^2$

or $(x+z).(x-z)=(p+q).(p-q)$.

To satisfy this equality, make

$$x+z=\frac{t}{v}(p+q), \text{ and } x-z=\frac{v}{t}(p-q)$$

which, by addition and subtraction, give

$$x=\frac{(t^2+v^2)p+(t^2-v^2)q}{2tv}$$

$$\text{and } z=\frac{(t^2-v^2)p+(t^2+v^2)q}{2tv}$$

But we have to make $x^2-z^2=p^2-q^2=\square$, which is the case when

$$p=2tv(d^2+e^2), \text{ and } q=2tv\times 2de;$$

hence substituting these values of p and q, we have

$$x=(t^2+v^2).(d^2+e^2)+(t^2-v^2).2de,$$

and $z=(t^2-v^2).(d^2+e^2)+(t^2+v^2).2de.$

Make $t^2+v^2=r$, and $t^2-v^2=s$,

then $x=r\ (d^2+e^2)+2des$, and $z=s(d^2+e^2)+2der$.

It now remains to make $z^2-y^2=q^2$, or $z^2-q^2=y^2$, viz.

$$\{s(d^2+e^2)+2der\}^2-16d^2e^2t^2v^2=\square.$$

This expression, by arranging the terms in the order of the exponents of d, becomes

$$s^2d^4+4rsed^3+(2e^2s^2+4r^2e^2-16e^2t^2v^2)d^2+4rse^3d+s^2e^4=\square,$$

which equate with

$$(sd^2+2red-se^2)^2=$$
$$s^2d^4+4rsed^3+(4r^2e^2-2e^2s^2)d^2-4rse^3d+s^2e^4,$$

and $(16e^2t^2v^2-4e^2s^2)d^2=8rse^3d$

Hence $d=\dfrac{2rse}{4t^2v^2-s^2}$,

from which solutions, *ad libitum*, may be had.

If $t=2$, $v=1$, and $e=7$; then $r=t^2+v^2=5$, $s=t^2-v^2=3$, $d=\dfrac{2rse}{4t^2v^2-s^2}=30$, and $q=2tv\times 2de=1680$; also

$$x=r(d^2+e^2)+2des=6005$$
$$z=s(d^2+e^2)+2der=4947$$
$$y=(z^2-q^2)^{\frac{1}{2}}=\qquad 4653,$$

the roots of three squares answering the proposed conditions.

Another Solution.

Let the numbers be denoted by x^2, y^2, and z^2; then, by the question, $x^2-y^2=a^2$, $x^2-z^2=b^2$, and $z^2-y^2=$ a square.

From the first two equations $x^2=a^2+y^2=b^2+z^2$.

Put $b=a+sv$, and $z=y-rv$,

then $a^2+y^2=b^2+z^2=(a+sv)^2+(y-rv)^2=$
$a^2+y^2+2asv-2rvy+s^2v^2+r^2v^2$

$$\therefore v=\frac{2ry-2as}{r^2+s^2}$$

and $z=y-rv=y-\dfrac{2r^2y-2ars}{r^2+s^2}=\dfrac{2ars-y(r^2-s^2)}{r^2+s^2}$.

Put $a=(r^2+s^2)2mn$, and $y=(r^2+s^2)(m^2-n^2)$ then will $x=(r^2+s^2)(m^2+n^2)$, and

$$z=4rsmn-(r^2-s^2)(m^2-n^2).$$

As z^2-y^2 is to be made a square, it is manifest, since $\frac{z+y}{2}\cdot\frac{z-y}{2}=\frac{z^2-y^2}{4}$, that if we can determine $\frac{z+y}{2}$ and $\frac{z-y}{2}$ each equal to a square, this condition will be fulfilled and the problem resolved.

$$\text{Now } \frac{z+y}{2}=s^2(m^2-n^2)+2rsmn=\Box$$

$$\text{and } \frac{z-y}{2}=r^2(n^2-m^2)+2rsmn=\Box.$$

Divide the first by s^2, and the second by r^2, also put $t=\frac{r}{s}$, then

$$m^2-n^2+2mnt=\Box$$

$$n^2-m^2+\frac{2mn}{t}=\Box.$$

Assume $m^2-n^2+2mnt=(qn-m)^2$

$$=q^2n^2-2qmn+m^2$$

and we shall have $n=2m\left(\frac{t+q}{q^2+1}\right)$, which, substituted in the other expression, gives

$$4m^2\left(\frac{t+q}{q^2+1}\right)^2-m^2+\frac{4m^2}{t}\left(\frac{t+q}{q^2+1}\right)$$

$$=\frac{m^2}{t^2(q^2+1)^2}\{4t^2(t+q)^2-t^2(q^2+1)^2+4t(t+q)(q^2+1)\}=\Box$$

$$\text{and } \therefore 4t^2(t+q)^2-t^2(q^2+1)^2+4t(t+q)(q^2+1)$$

$$=4t^4+8t^3q+t^2(-q^4+6q^2+3)+4tq(q^2+1)=\Box.$$

Assume its root $=2t^2+2tq-\frac{q^4-2q^2-3}{4}$, and then

$$4t^4+8t^3q+t^2(-q^4+6q^2+3)+4tq(q^2+1)$$

$$=4t^4+8t^3q+t^2(-q^4+6q^2+3)-tq(q^4-2q^2-3)+$$

$$\frac{1}{16}(q^4-2q^2-3)^2$$

which reduced, gives $t=\frac{r}{s}=\frac{(q^4-2q^2-3)^2}{16q(q^2+1)^2}$, in which q may be taken at pleasure.

If $q=1$, then $t=\frac{r}{s}=\frac{1}{4}$, and $n=2m\left(\frac{t+q}{q^2+1}\right)=\frac{5m}{4}$;

therefore $\frac{m}{n}=\frac{4}{5}$. Hence it appears that we may take $r=1$, $s=4$, $m=4$, and $n=5$,

Whence
$$\begin{aligned} x&=(r^2+s^2)(m^2+n^2)=697 \\ y&=(r^2+s^2)(m^2-n^2)=-153 \\ \text{and } z&=4rsmn-(r^2-s^2)(m^2-n^2)=185. \end{aligned}$$

Therefore 697, 185, and 153 are the roots of three squares that will answer.

Ex. 21. Let a^3 be the given cube, then

$$a^3-x^3-y^3 \text{ is to be a cube.}$$

Make $x=r-y$, and we have

$$a^3-3ry^2+3r^2y-r^3= \text{ a cube.}$$

Assume $(a-\frac{ry^2}{a^2})$ for its root; and we shall have $r=\frac{3a^3y}{a^3+y^3}$, and $x=r-y=\frac{y(2a^3-y^3)}{a^3+y^3}$.

If we take $a=2$, and $y=1$; then $x=\frac{5}{3}$, and $a^3-x^3-y^3=\left(\frac{4}{3}\right)^3$: so that the three cubes are 1, $\frac{64}{27}$, and $\frac{125}{27}$.

Ex. 22. Let x, y, and z be the roots of the required squares; then we have to find

$$\begin{aligned} x^2+y^2-z^2&=\square \\ x^2+z^2-y^2&=\square \\ y^2+z^2-x^2&=\square \end{aligned}$$

In which case, by first assuming

$$\begin{aligned} x&=p^2+q^2 \\ y&=p^2+pq-q^2 \\ z&=p^2-pq-q^2 \end{aligned}$$

we shall have

$$\begin{aligned} x^2+y^2-z^2&=(p^2-q^2+2pq)^2 \\ x^2+z^2-y^2&=(p^2-q^2-2pq)^2 \end{aligned}$$

Where the two first conditions being fulfilled, it only

remains to make a square of the third, which becomes, by substituting for x, y, and z,

$$y^2+z^2-x^2=p^4-4p^2q^2+q^4=\square$$

In order to reduce this to a more convenient form, let $p=(2+m)q$; then substituting this for p in the above equation, we have

$$y^2+z^2-x^2=q^4(m^4+8m^3+20m^2+16m+1)$$

where we have now to make the latter factor a square.

For which purpose, assume its root $=m^2+am+1$;

Then, by squaring, we have

$$\begin{array}{l} m^4+2am^3+(a^2+2)m^2+2am+1= \\ m^4+8m^3\ +20m^2\qquad\ +16m+1 \end{array}$$

Or, by making $2a=+16$,

$$16m^3+66m^2=8m^3+20m^2$$

Whence $46m^2=-8m^3$; or $m=-\dfrac{23}{4}$.

But $p=(2+m)q=-\dfrac{15}{4}q$; therefore we have $\dfrac{p}{q}=-\dfrac{15}{4}$;

whence, if $p=15$, and $q=-4$, we have

$$\begin{aligned} x&=p^2+q^2 &&=241 \\ y&=p^2+pq-q^2&&=149 \\ z&=p^2-pq-q^2&&=269 \end{aligned}$$

Consequently 241^2, 149^2, and 269^2 are squares, having the required conditions; and others might be found by giving different values to p and q.

Ex. 23. Let $(1+x)^3$, $(2-x)^3$, and y^3, be the required cubes; and let a be any given number; then by the question

$$\begin{aligned} (1+x)^3-a&=1+\ 3x+3x^2+x^3-a \\ (2-x)^3-a&=8-12x+6x^2-x^3-a \\ y^3-a&=\qquad\qquad\qquad\qquad y^3-a \end{aligned}$$

Whose sum $9-9x+9x^2+y^3-3a$ is to be a square.

Assume it $=(3x-r)^2=9x^2-6rx+r^2$,

And we shall have

$$9\quad 9x+y^3-3a=-6rx+r^2$$

Whence $x=\dfrac{r^2-y^3-9+3a}{6r\ -9}$,

where y and r may be assumed at pleasure, provided $6r$ be greater than 9, and r^2 be greater than y^3.

In our question $a=1$, and if we take $r=4$ and $y=2$,

$$\text{we shall have } x=\frac{2}{15};$$

$$\text{Whence } (1+x)^3=(1+\frac{2}{15})^3=\left(\frac{17}{15}\right)^3=\frac{4913}{3375}$$

$$(2-x)^3=(2-\frac{2}{15})^3=\left(\frac{28}{15}\right)^3=\frac{21952}{3375}$$

$$\text{and } y^3= \quad 2^3 \quad = \quad 8$$

Which are three cubes fulfilling the required conditions of the question.

Otherwise. Let x^3, y^3, b^3, be the three required cubes, and a the given number; then it is obvious that the question only requires that $x^3+y^3+b^3-3a=m^2$.

And here one of the cubes b^3 may be taken at pleasure; also a being given, we may consider $b^3-3a=c$, a given number; whence we have to find

$$x^3+y^3+c=m^2.$$

Let $\quad x=d+z$, and $y=f-z$,

Then $\quad x^3=d^3+3d^2z+3dz^2+z^3$

$\quad y^3=f^3-3f^2z+3fz^2-z^3$

By addition $d^3+f^3+3(d^2-f^2)z+3(d+f)z^2+c=m^2$

$\therefore 3(d+f)z^2+3(d^2-f^2)z+d^3+f^3+c=m^2$

Which may always be made a square, provided $3(d+f)=$ a square.

Assume $d=2$, and $f=1$, then $3(d+f)=9$, and the above becomes $9z^2+9z+9+c=m$

Let this $\quad =(3z-r)^2=9z^2-6rz+r^2$,

Then $\quad 9z+9+c=6rz+r^2$, and

$$z=\frac{r^2-9-c}{6r-9},$$

which is identical with the preceding expression.

SUMMATION AND INTERPOLATION
OF
INFINITE SERIES.

PROBLEM I.

Any series being given, to find the several orders of differences.

Ex. 3. Here 1, 3, 6, 10, 15, 21 given series,
2, 3, 4, 5, 6, 1st diff.
1, 1, 1, 1, 2d diff.

Ex. 4. Here 1, 6, 20, 50, 105, 196, &c. given series.
5, 14, 30, 55, 91, 1st diff.
9, 16, 25, 36, 2d diff.
7, 9, 11, 3d diff.
2, 2, &c. 4th diff.

Ex. 5. Here $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c. the given series.
$\frac{-1}{4}, \frac{-1}{8}, \frac{-1}{16}, \frac{-1}{32}$, &c. 1st diff.
$\frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c. 2d diff.
$\frac{-1}{16}, \frac{-1}{32}$, &c. 3d diff.
$\frac{1}{16}$, &c. 4th diff.
&c. &c.

PROBLEM II.

Any series, a, b, c, d, e, &c. being given, to find the first term of the nth order of differences.

Ex. 3. Here a, b, c, d, e, &c. are respectively
1, 3, 9, 27, 81, &c. also $n=8$.

Whence $a-nb+\frac{n(n-1)}{1.2}c-\frac{n(n-1)(n-2)}{1.2.3}d+$&c.

$=1-8b+28c-56d+70e-56f+28g-8h+i$
$=1-24+252-1512+5670-13608+$
$20412-17496+6561=32896-32640=256$

Ex. 4. Here $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{4}$, $d=\frac{1}{8}$, &c.

Also $n=5$; whence

$$-a+nb-\frac{n(n-1)}{1.2}c+\frac{n(n-1)(n-2)}{1.2.3}d-\&c.=$$

$$-a+5b-10c+10d-5e+f=$$

$$-1+\frac{5}{2}-\frac{10}{4}+\frac{10}{8}-\frac{5}{16}+\frac{1}{32}=$$

$$-\frac{122}{32}+\frac{121}{32}=-\frac{1}{32}\quad \text{Answer.}$$

PROBLEM III.

To find the nth term of the series, a, b, c, d, e, &c. when the differences of any order become at last equal to each other.

Ex. 3. Here 1, 4, 9, 16, 25, 36, given series,

3, 5, 7, 9, 11,	1st diff.
2, 2, 2, 2,	2d diff.
0, 0, 0,	3d diff.

Therefore $d'=3$, $d''=2$, and $n=15$;

Whence

$$a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1.2}d''$$

$$=1+14d'+91d''=1+42+182=225\quad \text{Answer.}$$

Ex. 4. Here 1, 8, 27, 64, 125, 216, given series.

7, 19, 37, 61, 91,	&c. 1st diff.
12, 18, 24, 30,	&c. 2d diff.
6, 6, 6,	&c. 3d diff.
0, 0,	&c. 4th diff.

Whence, since $d'=7$, $d''=12$, $d'''=6$, and $n=20$, we have

$$a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1.2}d''+\frac{(n-1)(n-2)(n-3)}{1.2.3}d'''$$

$$=1+19d'+171d''+969d'''$$

$$=1+133+2052+5814=8000\quad \text{Answer.}$$

Ex. 5. It will here be sufficient if we take merely the denominators of the proposed fractions, viz.

1, 3, 6, 10, 15,	&c. given series.
2, 3, 4, 5,	&c. 1st diff.
1, 1, 1,	&c. 2d diff.
0, 0,	&c. 3d diff.

Whence $d'=2$, and $d''=1$, also $n=30$,

$$\text{Therefore } a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1.2}d''=.$$

$$1+29d'+406d''=$$
$$1+58\quad+406\quad=465$$

Consequently $\frac{1}{465}$ is the 30th term sought.

PROBLEM IV.

To find the sum of n terms of series, a, b, c, d, e, &c. when the differences of any order become at last equal to each other.

Ex. 4. Here

2, 6, 12, 20, 30,	&c. given series.
4, 6, 8, 10,	&c. 1st diff.
2, 2, 2,	&c. 2d diff.
0, 0,	&c. 3d diff.

Therefore $a=2$, $d'=4$, $d''=2$;

$$\text{Consequently } na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''=$$

$$2n+\frac{n(n-1)}{1.2}\times 4+\frac{n(n-1)(n-2)}{1.2.3}\times 2=$$

$$2n^2+\frac{n(n^2-3n+2)}{3}=\frac{n(n+1)(n+2)}{3}\text{ Answer.}$$

Ex. 5. Here

1, 3, 6, 10, 15,	&c. given series.
2, 3, 4, 5,	&c. 1st diff.
1, 1, 1,	&c. 2d diff.
0, 0,	&c. 3d diff.

Consequently $a=1$, $d'=2$, $d''=1$,

Whence we have

$$na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''=$$

$$n+\frac{n(n-1)}{1.2}\times 2+\frac{n(n-1)(n-2)}{1.2.3}=$$

$$n^2+\frac{n(n^2-3n+2)}{6}=\frac{n(n+1)(n+2)}{1.2.3}\text{ Answer.}$$

Ex. 6. Here 1, 4, 10, 20, 35, &c. given series.

3, 6, 10, 15,	&c. 1st diff.
3, 4, 5,	&c. 2d diff.
1, 1,	&c. 3d diff.

Whence $a=1$, $d'=3$, $d''=3$, $d'''=1$; and therefore

$$na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''+$$

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d'''$$

$$=n+\frac{3n(n-1)}{2}+\frac{3n(n-1)(n-2)}{1.2.3}+$$

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$$

$$=\frac{3n^2-n}{2}+\frac{n(n^2-3n+2)}{2}+\frac{n(n^3-6n^2+11n-6)}{2.3.4}$$

$$=\frac{n(n^3+6n^2+11n+6)}{2.3.4}=\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}\text{ Answer.}$$

Ex. 7. Here 1, 16, 81, 256, 625, 1296, &c. given series.

15, 65, 175, 369, 671,	&c. 1st diff.
50, 110, 194, 302,	&c. 2d diff.
60, 84, 108,	&c. 3d diff.
24, 24,	&c. 4th diff.

Whence $a=1$, $d'=15$, $d''=50$, $d'''=60$, $d^{iv}=24$.

Substituting these in the general formula, we have

$$n+\frac{15n.(n-1)}{2}+\frac{50n.(n-1).(n-2)}{2.3}$$

$$+\frac{60n.(n-1).(n-2).(n-3)}{2.3.4}$$

$$+\frac{24n.(n-1).(n-2).(n-3).(n-4)}{2.3.4.5}=$$

$$\frac{1}{30}\times\left\{\begin{array}{r} +\ 30n \\ +225n^2-225n \\ +250n^3-750n^2+500n \\ +75n^4-450n^3+825n^2-450n \\ 6n^5-60n^4+210n^3-300n^2+144n \end{array}\right\}$$

$$\text{or } \frac{1}{30}(6n^5+15n^4+\ 10n^3\quad * \quad -\quad n)=$$

$$\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}, \text{ the answer.}$$

Example 8.—

Here 1, 32, 243, 1024, 3125, 7776, 16807, &c.	given series
31, 211, 781, 2101, 4651, 9031, &c.	1st diff.
180, 570, 1320, 2550, 4380, &c.	2d diff.
390, 750, 1230, 1830, &c.	3d diff.
360, 480, 600, &c.	4th diff.
120, 120, &c.	5th diff.

Whence $a=1$, $d'=31$, $d''=180$, $d'''=390$, $d^{iv}=360$ $d^{v}=120$. Substituting these in the general formula, we have

$$n+\frac{31n(n-1)}{2}+\frac{180n(n-1).(n-2)}{2.3}$$

$$+\frac{390n(n-1).(n-2).(n-3)}{2.3.4}$$

$$+\frac{360n(n-1).(n-2).(n-3).(n-4)}{2.3.4.5}$$

$$+\frac{120n(n-1).(n-2).(n-3).(n-4).(n-5)}{2.3.4.5.6}=$$

$$\frac{1}{12}\times\left\{\begin{array}{r} +\ 12n \\ +\ 186n^2-\ 186n \\ +\ 360n^3-1080n^2+\ 720n \\ +195n^4-1170n^3+2145n^2-1170n \\ +36n^5-360n^4+1260n^3-1800n^2+\ 864n \\ 2n^6-30n^5+170n^4-\ 450n^3+\ 548n^2-\ 240n \end{array}\right\}$$

or $\frac{1}{12}(2n^6+\ 6n^5+\ \ 5n^4\ \ \ *\ \ \ -n^2\ \ \ *\ \ )=$

$$\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}, \text{ the sum required.}$$

PROBLEM V

Any series a, b, c, d, e, &c. of equidistant terms being given, to find any intermediate term (z) by interpolation.

Ex. 2. In order to save the trouble of reduction, take the logarithms of the numbers; in which case we shall have

Terms.	*Logarithms.*	d',	d'',	d'''	d^{iv}
a, $\frac{1}{50}$	$=-1\cdot6989700$				
		-86002			
b, $\frac{1}{51}$	$=-1\cdot7075702$		$+1671$		
		-84331		-66	
c, $\frac{1}{52}$	$=-1\cdot7160033$		$+1605$		$+8$
		-82726		-58	
d, $\frac{1}{53}$	$=-1\cdot7242759$		$+1547$		
		-81179			
e, $\frac{1}{54}$	$=-1\cdot7323938$				

Where the 1st terms of the differences are $d' = -86002$, $d'' = +1671$, $d''' = -66$, $d^{\text{iv}} = +8$; and the distance of z, the term to be interpolated, being $3\frac{1}{2}$, we have

$$z = a + (x-1)d' + \frac{(x-1)(x-2)}{2}d'' + \frac{(x-1)(x-2)(x-3)}{2.3}d'''$$

$$+ \frac{(x-1)(x-2)(x-3)(x-4)}{2.3.4}d^{\text{iv}} + \&\text{c.}$$

$$= a + \frac{5}{2}d' + \frac{15}{8}d'' + \frac{5}{16}d''' - \frac{5}{128}d^{\text{iv}}$$

$$= -1{\cdot}6989700 - 215005 + 3133 - 21$$

$$= -1{\cdot}7201593 = \text{log. of } \frac{1}{52 \cdot 5} \text{ Answer.}$$

Ex. 3. Given the natural tangents of 88° 54′, 88° 55′, 88° 56′, 88° 57′, 88° 58′, 88° 59′, and 89°, to find that of 88° 58′ 18″.

Tangents.	d'	d''	d'''	d^{iv}	d^{v}
$a = 52{\cdot}080673$					
	801436				
$b = 52{\cdot}882109$		25042			
	826478		1193		
$c = 53{\cdot}708587$		26235		76	
	852713		1269		7
$d = 54{\cdot}561300$		27504		83	
	80217		1352		8
$e = 55{\cdot}441517$		28856		91	
	909073		1443		
$f = 56{\cdot}350590$		30299			
	939372				
$g = 57{\cdot}289962$					

Where the place of z being 5·3, we have

$$z = a + 4{\cdot}3d' + \frac{4{\cdot}3 \times 3{\cdot}3}{2}d'' + \frac{4{\cdot}3 \times 3{\cdot}3 \times 2{\cdot}3}{6}d''' +$$

$$\frac{4{\cdot}3 \times 3{\cdot}3 \times 2{\cdot}3 \times 1{\cdot}3}{24}d^{\text{iv}} + \frac{4{\cdot}3 \times 3{\cdot}3 \times 2{\cdot}3 \times 1{\cdot}3 \times {\cdot}3}{120}d^{\text{v}}$$

And consequently, by collecting the terms,

$$\begin{array}{r} 52\cdot080673 \\ 3\cdot446175 \\ 0\cdot177673 \\ 6489 \\ 134 \\ 1 \\ \hline 55\cdot711145 \end{array} = \text{tangent } 88^\circ\ 58'\ 18''$$

the answer required.

PROBLEM VI.

When the differences of any order of the series a, b, c, d, &c. are very small, or become equal to 0, to find any intermediate term.

Ex. 2. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50.

Here the number of terms are 5; and against 5 in the tablet we have $a-5b+10c-10d+5e-f=0$, or

$$f=a-5b+10c-10d+5e.$$

Num.	Roots.
$\sqrt[3]{45}$	$=3\cdot5568933=a$
$\sqrt[3]{46}$	$=3\cdot5830479=b$
$\sqrt[3]{47}$	$=3\cdot6088261=c$
$\sqrt[3]{48}$	$=3\cdot6342411=d$
$\sqrt[3]{49}$	$=3\cdot6593057=e$

Hence

$$\begin{array}{rr} a= & 3\cdot5568933 \\ 10c= & 36\cdot0882610 \\ 5e= & 18\cdot2965285 \\ \hline \text{Sum} & +57\cdot9416828 \\ \text{Sum} & -54\cdot2576505 \\ \hline & 3\cdot684032 \end{array} \qquad \begin{array}{rr} 5b= & 17\cdot9152395 \\ 10d= & 36\cdot3424110 \\ & \\ \hline & 54\cdot2576505 \end{array}$$

$3\cdot684032=$ the cube root of 50.

Ex. 3. Here, following the same process as before, we shall have

$$\begin{aligned}
a &= \text{log. } 50 = 1\cdot6989700 \\
b &= \text{log. } 51 = 1\cdot7075702 \\
c &= \text{log. } 52 = 1\cdot7160033 \\
d &= \text{log. } 53 \quad \textit{required.} \\
e &= \text{log. } 54 = 1\cdot7323938 \\
f &= \text{log. } 55 = 1\cdot7403627 \\
g &= \text{log. } 56 = 1\cdot7481880
\end{aligned}$$

Also the number of terms being 6, we have from the tablet

$$a-6b+15c-20d+15e-6f+g=0, \text{ or}$$

$$d=\frac{a-6b+15c+15e-6f+g}{20}$$

Whence, collecting the terms, we shall obtain

$$\begin{aligned}
a &= 1\cdot6989700 \\
15c &= 25\cdot7400495 \\
15e &= 25\cdot9859070 \\
g &= 1\cdot7481880 \\
\hline
\text{Sum } &+55\cdot1731145 \\
-6(b+f) &= 20\cdot6875974 \\
\hline
\text{Difference } &= 34\cdot4855171
\end{aligned}$$

Therefore $34\cdot4855171 \div 20 = 1\cdot72427586 =$ the log. of 53, as required.

EXAMPLES FOR PRACTICE.

Ex. 1. Here 2, 5, 8, 11, &c. given series
3, 3, 3, &c. 1st diff.

By Prob. IV. $n=100$, $a=2$, $d'=3$; and

$$s=na+\frac{n(n-1)}{1.2}d'=2\times100+\frac{100\times99}{2}\times3, \text{ or}$$

$s=200+14850=15050$, the sum required.

Ex. 2. By Prob. IV. Ex. 2, the sum

$$=\frac{n\times(n+1)\times(2n+1)}{6}=\frac{50\times51\times101}{6}=42925.$$

Ex. 3. Let $\frac{z}{(1-x)^3}=s=1+3x+6x^2+10x^3+\&c.$

Then $z=(1-x)^3\times(1+3x+6x^2+10x^3+\&c.)$

Which, by actual multiplication, is $=1$; therefore

$z=1$, and $s=\frac{1}{(1-x)^3}$, as required.

Ex. 4. Let $\frac{z}{(1-x)^4}=s=1+4x+10x^2+20x^3+35x^4+$&c.

Then $z=(1-x)^4\times(1+4x+10x^2+20x^3+35x^4+\text{\&c.})=1$ as appears by the actual operation;

Whence $z=1$, and $s=\frac{1}{(1-x)^4}$, the sum required.

Ex. 5. Let $z=\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+$ &c.

Then $z-1=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+$ &c.

And by subtraction,

$$1=\frac{2}{1.3}+\frac{2}{3.5}+\frac{2}{5.7}+\frac{2}{7.9}+\frac{2}{9.11}\text{ \&c.}$$
$$=2\left(\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}\text{ \&c.}\right)$$

Consequently by division,

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\text{ \&c.}=\frac{1}{2},$$

which is the sum required.

Ex. 6. The given series is equivalent to

2, 12, 30, 56, &c.
10, 18, 26, &c. 1st diff.
8, 8, &c. 2d diff.

By Prob. IV., $a=2$, $d'=10$, $d''=8$, $n=40$, and

$$s=na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2.3}d''$$
$$=80+780d'+9880d''$$
$$=80+7800+79040$$
$$=86920 \quad \text{Ans.}$$

Ex. 7. Here the given series

$$\frac{2x-1}{2x}+\frac{2x-3}{2x}+\frac{2x-5}{2x}+\frac{2x-7}{2x}+\text{ \&c.}$$

may be separated into the two

$$\frac{2x}{2x}+\frac{2x}{2x}+\frac{2x}{2x}+\frac{2x}{2x}+\frac{2x}{2x}+\text{\&c.}$$

$$-\frac{1}{2x}(1+3+5+7+9+\text{\&c.})$$

The sum of n terms of the former of which $=n$, and the same number of the latter $=\frac{n^2}{2x}$,

Whence $n-\frac{n^2}{2x}=\frac{2nx-n^2}{2x}=n\left(\frac{2x-n}{2x}\right)$, the sum required.

Ex. 8. Let $z=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\text{\&c.}$

Then $z-\frac{1}{6}=\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, by transposition;

Whence $\frac{1}{6}=\frac{3}{1.2.3.4}+\frac{3}{2.3.4.5}+\frac{3}{3.4.5.6}+\text{\&c.}$

by subtraction;

$$\therefore \frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\frac{1}{4.5.6.7}+\text{\&c.}=$$

$$\frac{1}{6}\div 3=\frac{1}{18}, \text{ the sum required.}$$

Ex. 9. Assume $z=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\text{\&c.}$

Then $z-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\text{\&c.}$

And by subtraction,

$$\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}+\text{\&c.}$$

Whence by division,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\text{\&c.}=\frac{1}{4}$$

Multiplying by 6, we get

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\&c.=\frac{3}{2}=1\frac{1}{2}$$

the sum required.

Ex. 10. Let $\frac{z}{(1-x)^4}=1+2^3x+3^3x^2+4^3x^3+5^3x^4+\&c.$

Then $z=(1-x)^4\times(1+2^3x+3^3x^2+4^3x^3+5^3x^4+\&c.)$

$$\text{Or}\begin{cases}(1+2^3x+3^3x^2+4^3x^3+\&c.)\times\\(1-4\ x+6\ x^2-4\ x^3+x^4)\end{cases}$$

$$z=\begin{cases}1+8x+27x^2+\ 64x^3+\&c.\\ \quad-4x-32x^2-108x^3-\&c.\\ \qquad\quad+\ 6x^2+\ 48x^3+\&c.\\ \qquad\qquad\quad-\quad 4x^3-\&c.\end{cases}$$

$$=1+4x+x^2+*+*+\&c.$$

Whence $\frac{1+4x+x^2}{(1-x)^4}=$ the sum required.

Ex. 11. Let $x=\frac{1}{r}$, and $s=\frac{z}{(r-1)^2}$;

$$\text{Then }\frac{z}{(r-1)^2}=\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}+\&c.$$

$$\text{And }z=(r-1)^2\times(\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\&c.)$$

Which, by the actual operation, becomes $=r$; hence $z=r$; and the sum of the series continued to infinity is

$$=\frac{r}{(r-1)^2}.$$

Now, for the other part, the terms after the nth are

$$\frac{n+1}{r^{n+1}}+\frac{n+2}{r^{n+2}}+\frac{n+3}{r^{n+3}}+\ \&c.\ ;\ \text{or}$$

$$\frac{1}{r^n}(\frac{n+1}{r}+\frac{n+2}{r^2}+\frac{n+3}{r^3}+\&c.)=$$

$$\frac{1}{r^n}\left\{\frac{n}{r}+\frac{n}{r^2}+\frac{n}{r^3}+\&c.+\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\&c.\right\}=$$

$$\frac{1}{r^n}\left\{\frac{n}{r-1}+\frac{r}{(r-1)^2}\right\}$$

Whence, by subtracting this from the whole sum before found, we have

$$\frac{r}{(r-1)^2}-\frac{1}{r^n}\left(\frac{n}{r-1}+\frac{r}{(r-1)^2}\right)=\frac{r}{(r-1)^2}-\frac{1}{r^n}\left\{\frac{nr+r-n}{(r-1)^2}\right\}=$$

the sum of n terms, as required.

Ex. 12. Let $z=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+$ &c.

Then $z-\frac{1}{2}-\frac{1}{4}=\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{14}+$ &c.

And therefore by subtraction,

$$\frac{3}{4}=\frac{4}{2.6}+\frac{4}{4.8}+\frac{4}{6.10}+\frac{4}{8.12}+\text{\&c.}$$

Whence $\Sigma=\frac{3}{16}=\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}+\frac{1}{8.12}+$ &c.

The infinite sum required.

Again, let $z=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\ldots\frac{1}{2n}$

Then $z-\frac{3}{4}=\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\ \ldots\ \frac{1}{2n}$

And $z-\frac{3}{4}+\frac{1}{2n+2}+\frac{1}{2n+4}=\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{14}$

$$+\text{\&c.}\ldots\frac{1}{2n+4};$$

Whence by subtraction,

$$\frac{3}{4}-\frac{1}{2n+2}-\frac{1}{2n+4}=\frac{4}{2.6}+\frac{4}{4.8}+\frac{4}{6.10}+\frac{4}{8.12}+\text{\&c.}$$

to n terms;

Consequently, by division,

$$\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}+\frac{1}{8.12}+\text{\&c. to } n \text{ terms.}$$

$$=\frac{3}{16}\quad\frac{1}{8(n+1)}-\frac{1}{8(n+2)}=\frac{3n^2+5n}{16n^2+48n+32}=z,$$

the sum of n terms, as required.

Ex. 13. Let $z=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\&c.$

Then $z-\frac{1}{3}=\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\&c.$

And by subtraction,

$$\frac{1}{3}=\frac{3}{3.6}+\frac{3}{6.9}+\frac{3}{9.12}+\frac{3}{12.15}+\&c.$$

Or $\frac{1}{3}=\frac{1}{2.3}+\frac{1}{3.6}+\frac{1}{4.9}+\frac{1}{5.12}+\&c.$

Hence $\frac{1}{3}\times\frac{1}{4}=\frac{1}{12}=\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}+\&c.$

And $\therefore\Sigma=\frac{1}{12}$, the sum to infinity.

Again, let $z=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\&c.\ldots\frac{1}{3n}$

Then $z-\frac{1}{3}=\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\&c.\ldots\frac{1}{3n}$

And $z-\frac{1}{3}+\frac{1}{3n+3}=\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\&c.\ldots\frac{1}{3n+3}$

Whence by subtraction,

$$\frac{1}{3}-\frac{1}{3(n+1)}=\frac{1}{3.2}+\frac{1}{6.3}+\frac{1}{9.4}+\frac{1}{12.5}+\&c.\ n\text{ terms.}$$

And dividing by 4,

$$\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}+\ \&c.\text{ to } n \text{ terms,}$$

$$=\frac{1}{12}-\frac{1}{12(n+1)}=\frac{n}{12(n+1)}=s,$$

the sum of n terms, as required.

Ex. 14. Let $z=\frac{1}{2}+\frac{1}{7}+\frac{1}{12}+\frac{1}{17}+\&c.$

Then $z-\frac{1}{2}=\frac{1}{7}+\frac{1}{12}+\frac{1}{17}+\frac{1}{22}+\&c.$

And by subtraction,

$$\frac{1}{2}=\frac{5}{2.7}+\frac{5}{7.12}+\frac{5}{12.17}+\frac{5}{17.22}+\&c.$$

Whence, multiplying by $\frac{6}{5}$, we have

$$\frac{6}{2.7}+\frac{6}{7.12}+\frac{6}{12.17}+\frac{6}{17.22}\&c. \text{ ad inf.} =\frac{3}{5}=\Sigma.$$

Now the general term of the series is

$$\frac{6}{(5n-3)(5n+2)},$$

And, therefore, to find the sum of all the terms beyond this, we need only assume

$$z=\frac{1}{5n+2}+\frac{1}{5n+7}+\frac{1}{5n+12}+\&c.$$

$$\text{Then } z-\frac{1}{5n+2}=\frac{1}{5n+7}+\frac{1}{5n+12}+\frac{1}{5n+17}+\&c.$$

And by subtraction,

$$\frac{1}{5n+2}=\frac{5}{(5n+2)(5n+7)}+\frac{5}{(5n+7)(5n+12)}+\&c.$$

Or multiplying by $\frac{6}{5}$, we have

$$\frac{6}{(5n+2)(5n+7)}+\frac{6}{(5n+7)(5n+12)}+\&c.=\frac{6}{5(5n+2)},$$

Which latter expression is the sum of all the terms after the nth; consequently

$$s=\frac{3}{5}-\frac{6}{5(5n+2)}=\frac{3n}{5n+2}, \text{ the sum of } n \text{ terms.}$$

Ex. 15. This series, by combining each positive term with the following negative, may be put under the form

$$\frac{1}{6}\left(\frac{5}{1.2.3.4}+\frac{9}{3.4.5.6}+\frac{13}{5.6.7.8}+\&c\right)$$

Let us, therefore, assume

$$z=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\&c.$$

$$\text{Then } z-\frac{1}{2}=\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\frac{1}{9.10}+\&c.$$

By subtraction, $\frac{10}{1.2.3.4}+\frac{18}{3.4.5.6}+\frac{26}{5.6.7.8}$, &c. $=\frac{1}{2}$;

Or by dividing by 2,

$$\frac{5}{1.2.3.4}+\frac{9}{3.4.5.6}+\frac{13}{5.6.7.8}+\&c.=\frac{1}{4}$$

and consequently

$$\frac{1}{6}\left(\frac{5}{1.2.3.4}+\frac{9}{3.4.5.6}+\frac{13}{5.6.7.8}+\&c.\right)=\frac{1}{24}, \text{ or}$$

$$\frac{1}{3.6}-\frac{1}{6.8}+\frac{1}{9.10}-\frac{1}{12.12}+\&c.=\frac{1}{24}=\Sigma.$$

Again, if the above assumed series be carried beyond the nth term, it becomes

$$\frac{1}{(2n+2)(2n+1)}+\frac{1}{(2n+4)(2n+3)}+\&c.$$

Whence as above $\frac{1}{12(2n+2)(2n+1)}=$ inf. sum, after n terms; consequently

$$\frac{1}{24}-\frac{1}{12(2n+2)(2n+1)}=\frac{n}{2(3+6n)}-\frac{n}{4(6+6n)}=$$

the sum of the series

$$\frac{1}{6}\left(\frac{5}{1.2.3.4}+\frac{9}{3.4.5.6}+\frac{13}{5.6.7.8}+\&c.\right)$$

to n terms. But as each term of this series is composed of two of the proposed, we must write $\frac{n}{2}$ for n in the expression

$$\frac{n}{2(3+6n)}-\frac{n}{4(6+6n)}, \text{ thus obtaining}$$

$$\frac{n}{4(3+3n)}-\frac{n}{8(6+3n)}$$

for the sum of the proposed series when n is even.

Again, for the sum when n is odd; if we write $n-1$ for n in the last expression, and add the nth term, viz. $\frac{1}{3n(4+2n)}$, the result will obviously be the sum in this case, that is

$$\frac{n-1}{12n}-\frac{n-1}{24(n+1)}+\frac{1}{3n(4+2n)}$$

for the sum of the given series when n is odd.

Ex. 16. Assume

$$z=\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\&c.$$

Then $z-\frac{1}{3}=-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\&c.$

Subtracting, $\frac{1}{3}=\frac{8}{3.5}-\frac{12}{5.7}+\frac{16}{7.9}-\frac{20}{9.11}+\frac{24}{11.13}+\&c.$

Dividing by 4,

$$\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\&c.=\frac{1}{12}=\Sigma.$$

Again, if the above assumed series be continued beyond the nth term, it will be

$$z=\pm\frac{1}{3+2n}\mp\frac{1}{5+2n}\pm\frac{1}{7+2n}\mp\frac{1}{9+2n}\&c.$$

And therefore, as before, $\pm\frac{1}{4(3+2n)}$ will be the infinite sum of the terms after the nth;

Consequently $\frac{1}{12}\pm\frac{1}{4(3+2n)}=$ sum of n terms required; the ambiguous sign being $+$ when n is odd, and $-$ when n is even.

Ex. 17. Assume

$$z=\frac{3}{1.2}+\frac{4}{2.3}+\frac{5}{3.4}+\frac{6}{4.5}+\&c.$$

Then $z-\frac{3}{2}=\frac{4}{2.3}+\frac{5}{3.4}+\frac{6}{4.5}+\frac{7}{5.6}+\&c.$

Subtracting, $\frac{3}{2}=\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{5.6.7}$ $+\&c.$ inf. sum.

Again, if the above assumed series be carried beyond the nth term, it will be

$$z=\frac{n+3}{(n+1)(n+2)}+\frac{n+4}{(n+2)(n+3)}+\frac{n+5}{(n+3)(n+4)},$$

Whence, as before, the first term

$$\frac{n+3}{(n+1)(n+2)}$$

will be the infinite sum of the terms after the nth.

Therefore

$$\frac{3}{2}-\frac{n+3}{(n+1)(n+2)}$$

or

$$\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}=$$

sum of n terms.

LOGARITHMS.

MULTIPLICATION BY LOGARITHMS.

Ex. 7.	Log. of 23·14 =	1·3643634
	log. of 5·062 =	0·7043221
	Prod. 117·1347	2·0686855
Ex. 8.	Log. of 4·0763 =	0·6102661
	log. of 9·8432 =	0·9931363
	Prod. 40·12383	1·6034024
Ex. 9	Log. of 498·256 =	2·6974525
	log. of 41·2467 =	1·6153892
	Prod. 20551·41	4·3128417

Ex. 10.	Log. of 4·26747 =	0·6301705
	log. of ·012345 =	$\bar{2}$·0914911
	Prod. ·05268191	$\bar{2}$·7216616
Ex. 11.	Log. of 3·12567 =	0·4949431
	log. of ·02868 =	$\bar{2}$·4575791
	log. of ·12379 =	$\bar{1}$·0926856
	Prod. ·01109706	$\bar{2}$·0452078
Ex. 12.	Log. of 2876·9 =	3·4589248
	·10674 =	$\bar{1}$·0283272
	·098762 =	$\bar{2}$·9945899
	·0031598 =	$\bar{3}$·4996596
	Prod. ·09583	$\bar{2}$·9815015

DIVISION BY LOGARITHMS.

Ex. 7.	Log. of 125 =	2·0969100
	log. of 1728 =	3·2375437
	Quot. ·0723379	$\bar{2}$·8593663
Ex. 8	Log. of 1728·95 =	3·2377825
	log. of 1·10678 =	0·0440613
	Quot. 1562·145	3·1937212
Ex. 9.	Log. of 10·23674 =	1·0101618
	log. of 4·96523 =	0·6959393
	Quot. of 2·061686	0·3142225
Ex. 10.	Log. of 19965·7 =	4·3002846
	log. of ·048235 =	$\bar{2}$·6833623
	Quot. 413926	5·6169223
Ex. 11.	Log. of ·067859 =	$\bar{2}$·8316075
	log. of 1234·59 =	3·0915229
	Quot. ·0000549648	$\bar{5}$·7400846

RULE OF THREE BY LOGARITHMS.

Ex. 5.	Comp. log. of 12·678	=	8·8969493
	log. of 14·065	=	1·1481397
	log. of 100·979	=	2·0042311
	Ans. 112·0263	=	2·0493201

Ex. 6.	Comp. log. of 1·9864	=	9·7019333
	log. of ·4678	=	$\bar{1}$·6700602
	log. of 50·4567	=	1·7029188
	Ans. 11·88262		1·0749123

Ex 7	Comp. log. of ·09658	=	11·0151128
	log. of ·24958	=	$\bar{1}$·3972098
	log. of ·008967	=	$\bar{3}$·9526472
	Ans. ·02317234		$\bar{2}$·3649698

Ex. 8.	Comp. log. of ·498621	=	$\overline{10}$·3022294
	log. of 2·9587	=	0·4711009
	log. of 2·9587	=	0·4711009
	3d prop. 17·55623		1·2444312

Comp. log. of 12·796	=	8·8929258
log. of 3·24718	=	0·5115064
log. of 3·24718	=	0·5115064
3d prop. ·8240216		$\bar{1}$·9159386

INVOLUTION BY LOGARITHMS.

Ex. 5.	Log. of 6·05987	=	0·7824633
			2
	Ans. 36·72203		1·5649266

Ex. 6.	Log. of ·176546 =	$\bar{1}$·2468579
		3
	Ans. ·005502674	$\bar{3}$·7405737
Ex. 7.	Log. of ·076543 =	$\bar{2}$·8839055
		4
	Ans. ·00003432591	$\bar{5}$·5356220
Ex. 8	Log. of 2·97643 =	0·4736957
		5
	Ans. 233·6031	2·3684785
Ex. 9.	Log. of 21·0576 =	1·3234089
		6
	Ans. 87187340	7·9404534
Ex. 10.	Log. of 1·09684 =	0·0401432
		7
	Ans. 1·909864	0·2810024

EVOLUTION BY LOGARITHMS.

Ex. 7.	Log. of 365·5674	2)2·5629675
	Ans. 19·11981	1·2814837
Ex. 8.	Log. of 2·987635	3)0·4753276
	Ans. 1·440265	0·1584425
Ex. 9.	Log. of ·967845	4)$\bar{1}$·9858059
	Ans. ·9918624	$\bar{1}$·9964515
Ex. 10	Log. of ·098674	7)$\bar{2}$·9942027
	Ans. ·7183146	$\bar{1}$·8563147

Ex. 11. Log. of 21 $= 1 \cdot 3222193$
log. of 373 $= 2 \cdot 5717088$

$$\begin{array}{r} \bar{2}\cdot 7505105 \\ 2 \\ \hline 3)\bar{3}\cdot 5010210 \\ \hline \text{Ans. } \cdot 146895 \quad \bar{1}\cdot 1670070 \end{array}$$

Ex. 12. Log. of 112 $= 2 \cdot 0492180$
log. of 1727 $= 3 \cdot 2372923$

$$\begin{array}{r} \bar{2}\cdot 8119257 \\ 3 \\ \hline 5)\bar{4}\cdot 4357771 \\ \hline \text{Ans. } \cdot 1937115 \quad \bar{1}\cdot 2871554 \end{array}$$

MISCELLANEOUS EXAMPLES.

Ex. 1. Log. of 2 $= 0 \cdot 3010300$
log. of 123 $= 2 \cdot 0899051$

$$\begin{array}{r} 2)\bar{2}\cdot 2111249 \\ \hline \text{Ans. } \cdot 1275153 \quad \bar{1}\cdot 1055624 \end{array}$$

Ex. 2. Comp. log. of $3 \cdot 14159$ $\quad 3)\bar{1}\cdot 5028505$

$$\begin{array}{r} \text{Ans. } \cdot 6827842 \quad \bar{1}\cdot 8342835 \end{array}$$

Ex. 3. Log. $\cdot 00563 = \bar{3}\cdot 7505084$
$\cdot 07 = \frac{7}{100}$, theref. 7

$$\begin{array}{r} 100)\overline{16}\cdot 2535588 \\ \hline \text{Ans. } \cdot 6958818 \quad \bar{1}\cdot 8425355 \end{array}$$

The student will observe, that 84 is borrowed in this example to make the 16 up to 100, according to the rule.

Ex. 4. $$\frac{(\frac{2}{3})^{\frac{1}{2}} \times (\frac{3}{4})^{\frac{1}{3}}}{17\frac{1}{3}} = \left(\frac{8}{27} \times \frac{9}{16}\right)^{\frac{1}{6}} \div \frac{52}{3} = \left(\frac{1}{6}\right)^{\frac{1}{6}} \times \frac{3}{52}.$$

Comp. log. of 6	6)$\bar{1}$·2218487
	$\bar{1}$·8703081
log. of 3	0·4771213
comp. log. of 52	$\bar{2}$·2839967
Ans. ·04279826	$\bar{2}$·6314261

Ex. 5.

Log. 5	=	0·6989700
log. 8	=	0·9030900
		2)$\bar{1}$·7958800
$\therefore$ log. $(\frac{5}{8})^{\frac{1}{2}}$	=	$\bar{1}$·8979400
log. 7	=	0·8450980
log. 11	=	1·0413927
		3)$\bar{1}$·8037053
$\therefore$ log. $(\frac{7}{11})^{\frac{1}{3}}$	=	$\bar{1}$·9345684
log. ·012	=	$\bar{2}$·0791812
log. $(\frac{5}{8})^{\frac{1}{2}}$	=	$\bar{1}$·8979400
comp. log. 7	=	$\bar{1}$·1549020
Ans. ·001165713		$\bar{3}$·0665916

Ex. 6.

Log 11	=	1·0413927
log. 21	=	1·3222193
		2)$\bar{1}$·7191734
$\therefore$ log. $\sqrt{\frac{11}{21}}$	=	$\bar{1}$·8595867
comp. log. 9	=	$\bar{1}$·0457575
$\therefore$ log. $\frac{1}{9}\sqrt{\frac{11}{21}}$	=	$\bar{2}$·9053442

$$
\begin{aligned}
\text{log. } 15{\cdot}2 &= 1{\cdot}1818436 \\
\therefore \text{log. } \sqrt[3]{15\tfrac{1}{5}} &= 0{\cdot}3939479 \\
\text{log. } {\cdot}03 &= \bar{2}{\cdot}4771213 \\
\therefore \text{log. } ({\cdot}03\sqrt[3]{15\tfrac{1}{5}}) &= \bar{2}{\cdot}8710692 \\
\text{also log. } \tfrac{1}{9}\sqrt{\tfrac{11}{21}} &= \bar{2}{\cdot}9053442 \\
\therefore \text{log. of numerator} &= \bar{3}{\cdot}7764134
\end{aligned}
$$

$$
\begin{aligned}
\text{log. } 12{\cdot}2 &= 1{\cdot}0863598 \\
\therefore \text{log. } \sqrt[3]{12\tfrac{1}{5}} &= 0{\cdot}3621199
\end{aligned}
$$

$$
\text{log. } 7\tfrac{1}{3} \begin{cases} \text{log. } 22 = 1.3424227 \\ \text{comp. log. } 3 = \bar{1}{\cdot}5228787 \end{cases}
$$

$$
\therefore \text{log. } 7\tfrac{1}{3}\sqrt[3]{12\tfrac{1}{5}} = 1{\cdot}2274213
$$

$$
\begin{aligned}
\text{log. } 17{\cdot}125 &= 1{\cdot}2336306 \\
\therefore \text{log. } \sqrt[4]{17\tfrac{1}{8}} &= 0{\cdot}3084077 \\
\text{log. } {\cdot}19 &= \bar{1}{\cdot}2787536 \\
\therefore \text{log. } {\cdot}19\sqrt[4]{17\tfrac{1}{8}} &= \bar{1}{\cdot}5871613 \\
\text{also log. } 7\tfrac{1}{3}\sqrt[3]{12\tfrac{1}{5}} &= 1{\cdot}2274213 \\
\therefore \text{log. of denom}^{\text{r}}. &= 0{\cdot}8145826 \\
\text{comp. log. of denom}^{\text{r}}. &= \bar{1}{\cdot}1854174 \\
\text{log. of num}^{\text{r}}. &= \bar{3}{\cdot}7764134 \\
\therefore \text{log. of fraction} &= \bar{4}{\cdot}9618308
\end{aligned}
$$

$$
\therefore \text{fraction} = {\cdot}0009158636 \quad \text{Ans.}
$$

Ex. 7.

$$
\begin{aligned}
\text{Log. } 19 &= 1{\cdot}2787536 \\
\therefore \text{log. } \sqrt{19} &= 0{\cdot}6393768 \\
\text{log. } 5 &= 0{\cdot}6989700 \\
\text{comp. log. } 6 &= \bar{1}{\cdot}2218487 \\
\therefore \text{log. } \tfrac{5}{6}\sqrt{19} &= 0{\cdot}5601955 \\
\text{and } \tfrac{5}{6}\sqrt{19} &= 3{\cdot}632415
\end{aligned}
$$

$$\begin{aligned}
\log. 35\tfrac{1}{3} &= 1\cdot5481846 \\
\therefore \sqrt[3]{\ } \log. 35\tfrac{1}{3} &= 0\cdot5160615 \\
\log. 4 &= 0\cdot6020600 \\
\text{comp. log. } 7 &= \bar{1}\cdot1549020 \\
\therefore \log. \tfrac{4}{7}\sqrt[3]{35\tfrac{1}{3}} &= 0\cdot2730235 \\
\text{and } \tfrac{4}{7}\sqrt[3]{35\tfrac{1}{3}} &= 1\cdot875096 \\
\text{but } \tfrac{5}{6}\sqrt{19} &= 3\cdot632415 \\
\therefore \tfrac{5}{6}\sqrt{19}+\tfrac{4}{7}\sqrt[3]{35\tfrac{1}{3}} &= 5\cdot507511 \ \ldots (1)
\end{aligned}$$

$$\begin{aligned}
\log. 28\tfrac{2}{3} &= 1\cdot4573772 \\
\therefore \log. \sqrt{28\tfrac{2}{3}} &= 0\cdot7286886 \\
\log. 15 &= 1\cdot1760913 \\
\text{comp. log. } 11 &= \bar{2}\cdot9586073 \\
\therefore \log. \tfrac{15}{11}\sqrt{28\tfrac{2}{3}} &= 0\cdot8633872 \\
\text{and } \tfrac{15}{11}\sqrt{28\tfrac{2}{3}} &= 7\cdot301082 \\
\text{but } 14\tfrac{7}{19} &= 14\cdot368421 \\
\therefore 14\tfrac{7}{19}-\tfrac{15}{11}\sqrt{28\tfrac{2}{3}} &= 7\cdot067339 \ \ldots (2)
\end{aligned}$$

$$\begin{aligned}
\log. (1) &= 0\cdot7409554 \\
\text{comp. log. } (2) &= \bar{1}\cdot1507440 \\
\log. 127 &= 2\cdot1038037 \\
\text{comp. log. } 4 &= \bar{1}\cdot3979400 \\
& 1\cdot3934431
\end{aligned}$$

the natural number answering to which is $24\cdot7447$, the answer.

MISCELLANEOUS QUESTIONS.

Ex. 1. Let x be the number of minutes after 8, or the number the minute hand is before it overtakes the hour hand, after the former is at 12, and the latter at 8.

Then $\dfrac{x}{12}$ will be the number of minutes that the hour hand has advanced in the same time. And by the question

$$40+\frac{1}{12}x=x, \text{ or } \frac{11}{12}x=40, \text{ or}$$

$$x=\frac{40\times 12}{11}=\frac{480}{11}=43\frac{7}{11};$$

Therefore the time was 8h. 43m. $38\frac{2}{11}$ sec.

Ex. 2. Let x represent the digit in the place of 10's, and y that in the units; then will $10x+y=$ the number itself, and $10y+x$ the number formed by the inverted digits. Hence, by the question,

$$\left.\begin{array}{l} x^2-y^2=10x+\quad y \\ 10x+y+36\ =10y+x \end{array}\right\}$$

From the latter we have $9x-9y=-36$, or $x\quad y=-4$, or $y=x+4$; whence it appears that y is greater than x, and our first equation becomes

$$y^2-x^2=10x+y;$$

By substituting the above value of y, we have

$$(x+4)^2-x^2=10x+x+4$$

That is $8x+16=10x+x+4$, or

$3x=12$, or $x=4$, and $y=x+4=8$;

Consequently $10x+y=(10\times 4)+8=48$, the number sought.

Ex. 3. Let x and y represent the two numbers; then by the question

$$\left.\begin{array}{l} x-y : x+y :: 2 : 3 \\ x+y : \quad xy \quad :: 3 : 5 \end{array}\right\} \text{or}$$

$$\left.\begin{array}{l} 3x-3y=2x+2y \\ 5x+5y=3xy \end{array}\right\}$$

From the former of these $x=5y$, which substituted for x in the latter, gives

$$25y+5y=15y^2, \text{ or } 30y=15y^2$$

Whence, dividing by $15y$, we have $y=2$, and consequently $x=5y=10$.

Ex. 4. Let x be the number of games won, and y the number lost; then by the question

$$\left.\begin{array}{l} x+\ y=20 \\ 2x-3y=\ 5 \end{array}\right\}$$

Multiply the first by 3, and we have

$$3x+3y=60$$

Add the latter $2x-3y=\ 5$

and we have $5x \quad =65$, or $x=13$, games won.

And consequently $20-x=20-13=7$, games lost.

Ex. 5. Let x be the number of yards in the four sides; then $3x$ is the number of feet; and therefore, by the question, $3x-150$ is the number of palisades,

As is also $x+70$; and consequently $3x-150=x+70$, $\therefore 2x=220$, or $x=110$;

Therefore $x+70=180$, the number sought.

Ex. 6. Let x be the number of hours in which B will fill it; then $\frac{1}{x}$ will be the quantity thrown in by B in an hour, and $\frac{1}{20}$ is the quantity thrown in by A in an hour.

Also since the two together will fill it in 12 hours, $\frac{1}{12}$ will be the quantity the two throw in an hour.

Whence $\frac{1}{x}+\frac{1}{20}=\frac{1}{12}\therefore\frac{1}{x}=\frac{1}{12}-\frac{1}{20}=\frac{1}{30}$, or $x=30$ Ans.

Ex. 7. Let $x=$ number of pounds of tea, and $y=$ number of pounds of coffee,

Then $10x+2\cdot5y=625$, and $8\,x+4\cdot5y=725$.

From the first of which equations we have

$x=\frac{625-2\cdot5y}{10}$, and from the second $x=\frac{725-4\cdot5y}{8}$

$$\text{Consequently } \frac{625-2\cdot5y}{10}=\frac{725-4\cdot5y}{8},$$

$$\therefore 5000-20y=7250-45y,$$

$$\text{Whence } 25y=2250, \text{ or } y=\frac{2250}{25}=90, \text{ and}$$

$$x=\frac{625-(2\cdot5\times90)}{10}=\frac{625-225}{10}=\frac{400}{10}=40$$

Therefore 90 pounds of coffee, and 40 pounds of tea are the required quantities.

Ex. 8. Let x be the required number; then by the question

$$(x+3) : (x+19) :: (x+19) : (x+51)$$

Since the product of the means and extremes are equal to each other, we have

$$(x+3)(x+51)=(x+19)^2, \text{ or}$$
$$x^2+54x+153=x^2+38x+361,$$
$$\therefore 54x-38x=208, \text{ or}$$
$$16x=208, \text{ or } x=13 \text{ Ans.}$$

Ex. 9. Let r be the ratio; then, since 3 is the first term, $3 : 3r :: 3r^2 : 24$ are the first four proportionals, and $3r$ and $3r^2$ the two means sought.

Hence, since the product of the extremes and means are equal, we have

$$9r^3=72, \text{ or } r^3=8, \text{ or } r=2;$$

and therefore 6 and 12 are the means required.

In the same way, in the second part, we have

$$3 : 3r :: 3r^2 : 3r^3 :: 3r^4 : 96,$$

Whence also we have $3r\times 3r^4=3\times 96$, or

$$9r^5=288, \text{ or } r^5=32, \text{ or } r=2,$$

Therefore 6, 12, 24, and 48, are the four mean proportionals.

Ex. 10. Let x, rx, r^2x, r^3x, r^4x, r^5x, be the six proportionals.

Then by the question

$$\left.\begin{aligned} x+rx+r^2x+r^3x+r^4x+r^5x&=315 \\ \text{and } x \qquad\qquad\qquad\qquad +r^5x&=165 \end{aligned}\right\}$$

$$\text{By subtraction} \quad rx+r^2x+r^3x+r^4x=150$$

Now, by the rule for summing a geometrical progression, our first equation may be written

$$\left.\begin{aligned} \left(\frac{r^6-1}{r-1}\right)x&=315 \\ \text{and the last } \left(\frac{r^4-1}{r-1}\right)rx&=150 \end{aligned}\right\}$$

Whence, dividing the former of these by the latter, &c.

$$\frac{r^6-1}{r^4-1}=\frac{315r}{150}=\frac{21r}{10}$$

Also, reducing this by dividing both numerator and denominator by r^2-1, we have

$$\frac{r^4+r^2+1}{r^2+1}=\frac{21r}{10}, \text{ or}$$

$$r^4+r^2+1=\frac{21r}{10}(r^2+1)$$

And by adding r^2 to both sides

$$r^4+2r^2+1=\frac{21r}{10}(r^2+1)+r^2, \text{ or}$$

$$(r^2+1)^2-\frac{21r}{10}(r^2+1)=r^2$$

Hence by completing the square

$$(r^2+1)^2-\frac{21r}{10}(r^2+1)+\frac{441r^2}{400}=r^2+\frac{441r^2}{400}=\frac{841r^2}{400}$$

And by extracting the root $(r^2+1)-\frac{21}{20}r=\frac{29r}{20}$

$$\text{Or } r^2-\frac{25}{10}r=-1$$

$$\text{Or } r=\frac{25}{20}+\sqrt{(\frac{625}{400}-1)}=\frac{25}{20}+\frac{15}{20}=2\,;$$

But from our first reduced equation we have

$$x=\frac{315(r-1)}{r^6-1}=\frac{315}{63}=5\,;$$

Therefore 5, 10, 20, 40, 80, and 160, are the proportionals sought.

Ex. 11. Let $x=$ length, and $y=$ breadth of the rectangle.

Then, because $2x+2y=$ perimeter both of the rectangle and the square, we shall have

$$\frac{2x+2y}{4}, \text{ or } \frac{x+y}{2}= \text{ side of the square,}$$

But $xy=2$, and $\frac{x^2+2xy+y^2}{4}=4$, or $x^2+2xy+y^2=16$, by the question,

Whence, since $xy=2$, or $y=\frac{2}{x}$, we shall also have

$$x^2+4+\frac{4}{x^2}=16, \text{ or } x^4-12x^2=-4 \text{; which gives}$$

$$x^2=6+4\sqrt{2}, \text{ or } x=\sqrt{(6+4\sqrt{2})}=2+\sqrt{2};$$

$$\text{And therefore } y=\frac{2}{2+\sqrt{2}}=2-\sqrt{2},$$

multiplying each of which by $\sqrt{160}$, the number of poles in an acre, we shall obtain $x=43 \cdot 1868$, and $y=7 \cdot 4097$.

Ex. 12. Let x be the number of men employed at first; then $x+16$ will be the number in the second instance; and since the time in performing an equal quantity of work is reciprocally as the number of men employed, we have

$$x : 16 :: x+16 : 24$$

Whence, $24x=16x+256$, or $x=\frac{256}{8}=32$, the number of men at first, and $32+16=48$, the number during the second part of the time.

$$\text{Hence } \left.\begin{aligned} 32\times 24\times 1\tfrac{1}{2}&=1152s. \\ 48\times 16\times 1\tfrac{1}{2}&=1152s. \end{aligned}\right\}=115l.\ 4s. \text{ Answer.}$$

Ex. 13. Let x and y be the two numbers; then by the question, we have

$$\left.\begin{aligned} x^2+y&=\ 62 \\ y^2+x&=176 \end{aligned}\right\}$$

From the first $y=62-x^2$, and consequently

$$y^2=62^2-124x^2+x^4$$

This substituted in the second gives

$$62^2-124x^2+x^4+x=176$$

$$\text{or, } x^4-124x^2+x=-3668$$

Where x will be found $=7$, and consequently $y=62-x^2=13$; but the question cannot, I believe, be in any manner reduced to a quadratic form, at least while it is considered under the general form

$$x^2+y=a, \text{ and } y^2+x=b.$$

Ex. 14. Let x denote the number of feet in the circumference of the less wheel, and y the number in the circumference of the greater wheel.

Then will $\frac{360}{x}$ be the number of revolutions of the less wheel, and $\frac{360}{y}$ the number of revolutions of the greater.

Whence by the question $\begin{cases} \frac{360}{x} - \frac{360}{y} = 6 \\ \frac{360}{x+3} - \frac{360}{y+3} = 4 \end{cases}$

From the first of which, $360(y-x)=6xy$
From the second, $360(y-x)=4(x+3)(y+3)$
The latter of which by multiplication gives

$$360y-360x=4xy+12x+12y+36$$
$$\text{or } 348y-372x=4xy+36$$

And the former, by division,

$$60y-\ 60x=\ xy$$

Mult. this by 4, $240y-240x=4xy$
Subtract $348y-372x=4xy+36$
And we have $-108y+132x=-36$

Whence $x=\frac{108y-36}{132}=\frac{9y-3}{11}$

Substitute this value of x in the equation

$$60y-60x=xy,$$

And we have

$$60y-\frac{540y-180}{11}=\frac{9y^2-3y}{11}, \text{ or}$$
$$660y-540y+180=9y^2-3y, \text{ or}$$
$$9y^2-123y=180, \text{ or}$$
$$y^2-\frac{41}{3}y=20; \text{ whence}$$
$$y=\frac{41}{6}+\sqrt{(\frac{1681}{36}+20)}=\frac{41}{6}+\frac{49}{6}=\frac{90}{6}=15$$

and consequently $x=\frac{9y-3}{11}=\frac{135-3}{11}=12$; that is, the circumference of the greater wheel $=15$, and the circumference of the less $=12$.

Ex. 15. Let x=number of sheep he bought;

then $\frac{40x}{3}+\frac{36x}{4}+34(x-\frac{x}{3}-\frac{x}{4}) = 36\frac{1}{2}x =$ the money they sold for.

Whence $36\frac{1}{2}x - 10l.\ 14s. = 98l.\ 16s.$, or $36\frac{1}{2}x - 214 = 1976$ by the question,

Therefore, by reduction, $73x = 4380$, and $x = \frac{4380}{73} = 60$ Answer.

Ex. 16. Let x=B's share; then $2x$=A's share, and $3x$=C's share; therefore $6x$= sum of all their shares, which must $=300l.$; hence, $x=50l.$ B's share, $2x=100l.$ A's share, and $3x=150l.$ C's share.

Ex. 17. Let x represent the number of persons, and y the number of pounds each received, then xy is the whole sum divided.

Now by the question

$$\left.\begin{array}{r}(x-3)\times(y+150)=xy\\(x+6)\times(y-120)=xy\end{array}\right\}\text{ or}$$

$$\left.\begin{array}{r}xy-3y+150x-450=xy\\xy+6y-120x-720=xy\end{array}\right\}\text{ or}$$

$$\left.\begin{array}{r}150x-3y=450\\-120x+6y=720\end{array}\right\}$$

Multiply the first equation by 2, and we have

$$300x-6y=900$$

Add the 2d $\qquad -120x+6y=720$

Whence $180x=1620$, or $x=9$, the number of persons;

And consequently $y=\frac{300x-900}{6}=300l.$, the sum each received; and $9\times300=2700l.$ the whole sum divided.

Ex. 18. Let x= original value of each piece,

Then $x-\frac{x}{5}$, or $\frac{4x}{5}$= value of each, after they had been filed.

But, by the question $16\times\frac{4x}{5}$, or $\frac{64x}{5}=11l.\ 4s.=224s.$

Whence $x=\frac{224\times5}{64}=17s.\ 6d.$ Answer.

Ex. 19. Let x be the number of cubic inches of tin, and y of copper.

Then by the question,

$$\left.\begin{aligned} x+\quad y&=100 \\ 4\tfrac{1}{4}x+5\tfrac{1}{4}y&=505 \end{aligned}\right\}$$

Multiply the second equation by 4, and the first by 17; then

$$17x+21y=2020$$
$$17x+17y=1700$$

And by subtraction $4y=320$, or $y=80$, and $x=100-y=20$; therefore $4\frac{1}{4}x=85$ the ounces of tin, and consequently $505-85=420$ ounces of copper.

Ex. 20. Let x and $500-x=$ the sums they respectively advanced.

Then $500:160$, or $25:8::x:\frac{8x}{25}=$ A's share.

And $25:8::500-x:\frac{4000-8x}{25}=$ B's share.

But by the question $\frac{8x}{25}-32=\frac{4000-8x}{25}$, or

$$8x-800=4000-8x;$$

whence $16x=4800$, or $x=\frac{4800}{16}=300$; and $500-300=200$. Answer.

Ex. 21. Let x represent the number of $7s.$ pieces, and y the number of dollars.

Then by the question

$$7x+4\tfrac{1}{2}y=2000 \text{ shillings, or}$$
$$14x+9\ y=4000$$

Consequently $y=\frac{4000-14x}{9}=$ whole number.

$$\text{Or } y=444-x+\frac{4-5x}{9}=wh.$$

Let now $\frac{4-5x}{9}=p$; then $9p=4-5x$, or

$$x=\frac{4-9p}{5}=1-2p-\frac{1-p}{5}$$

Assume $\frac{1-p}{5}=q$, and we have $p=-5q+1$.

Consequently $x=\frac{45q-5}{5}=9q-1$, where q may be taken at pleasure; if we take

$$q=1,\ \ 2,\ \ 3,\ \ 4,\ \ 5,\ \&c.$$
$$x=8,\ 17,\ 26,\ 35,\ 44,\ \&c.$$

But the greatest value of x cannot exceed $\frac{4000-9}{14}$ $=285$; therefore $\frac{285}{9}=31$, the number of different ways.

Otherwise. By the rule before given, we find from the equation $14p-9q=1$

$$\therefore p=2 \text{ and } q=3;$$

$$\text{whence } \frac{2\times 4000}{9}-\frac{3\times 4000}{14}=31,$$

the number of different ways; the same as before.

Ex. 22. Let x and y represent the two numbers; then by the question

$$\left.\begin{aligned} x+y&=2 \\ x^9+y^9&=32 \end{aligned}\right\}$$

Assume $xy=p$; then

$$x^3+y^3=(x+y)^3-3xy(x+y)=8-6p$$
$$\begin{aligned} \text{and } x^9+y^9&=(x^3+y^3)-3x^3y^3(x^3+y^3) \\ &=(8-6p)^3-3p^3(8-6p) \\ &=512-1152p+864p^2-240p^3+18p^4 \end{aligned}$$
$$\therefore\ 18p^4-240p^3+864p^2-1152p+512=32$$
$$\text{or } p^4-\frac{40}{3}p^3+48p^2-64p+\frac{80}{3}=0$$

Which equation is resolvable into the factors

$$(p^2-4p+4)(p^2-\frac{28}{3}p+\frac{20}{3})=0$$

Whence, by the solution of these two quadratics, we have

$$p=2,\ p=2,\ p=\frac{14}{3}+\frac{\sqrt{136}}{3},\ \text{and}\ p=\frac{14}{3}-\frac{\sqrt{136}}{3}$$

But $x+y=2$, and $xy=p$;

Whence $x-y=\pm\sqrt{(4-4p)}=\pm 2\sqrt{(1-p)}$

Or $x=1\pm\sqrt{(1-p)}$, and $y=1\mp\sqrt{(1-p)}$;

And substituting here the above values of p, we have the following solutions, viz.

$$x=1\pm\sqrt{-1},\ \text{and}\ y=1\mp\sqrt{-1}$$

$$x=1\pm\sqrt{\left\{-\frac{11}{3}-\frac{2\sqrt{34}}{3}\right\}},\ \text{and}$$

$$y=1\mp\sqrt{\left\{-\frac{11}{3}-\frac{2\sqrt{34}}{3}\right\}}$$

$$x=1\pm\sqrt{\left\{-\frac{11}{3}+\frac{2\sqrt{34}}{3}\right\}},\ \text{and}$$

$$y=1\mp\sqrt{\left\{-\frac{11}{3}+\frac{2\sqrt{34}}{3}\right\}}$$

Where it may be observed, that the two latter of these are the only real answers; the others being imaginary.

Ex. 23. For a Solution, see Ex. 14, page 70, Quadratic Equations.

Ex. 24. Let x and y be the two numbers; then

The geometrical mean $=\sqrt{xy}$

The arithmetical mean $=\frac{1}{2}x+\frac{1}{2}y$

The harmonical mean $=\dfrac{2xy}{x+y}$

Therefore by the question

$$\tfrac{1}{2}x+\tfrac{1}{2}y-\sqrt{xy}=13$$

$$\sqrt{xy}-\frac{2xy}{x+y}=12.$$

From the first of which equations we have

$$x+y=26+2\sqrt{xy}$$

And from the 2d, $x+y=\dfrac{2xy}{\sqrt{xy}-12}$

Consequently $26+2\sqrt{xy}=\dfrac{2xy}{\sqrt{xy}-12}$;

Whence $26\sqrt{xy}+2xy-312-24\sqrt{xy}=2xy$

Or $2\sqrt{xy}=312$, or $\sqrt{xy}=156$, and $xy=24336$;

Therefore by substituting the value of $\sqrt{xy}$ in the equation $x+y=26+2\sqrt{xy}$, and repeating our last, we have

$$x+y=338$$
$$xy=24336$$

By squaring the first of these

$$x^2+2xy+y^2=114244$$

also $\quad 4xy \quad = \quad 97344$

By subtract. $\quad x^2-2xy+y^2=16900$

By extraction $\quad x-y=130$

Also $\quad x+y=338$;

Whence by addition $\quad 2x=468$, or $x=234$

And by subtraction $\quad 2y=208$, or $y=104$.

Ex. 25. Here $\left.\begin{array}{l} x^3y+xy^3=3 \\ x^6y^2+x^2y^6=7 \end{array}\right\}$

By squaring the first equation, we have

$$x^6y^2+2x^4y^4+x^2y^6=9$$

Subt. the 2d, $\quad x^6y^2 \qquad +x^2y^6=7$

$$2x^4y^4=2, \text{ or } xy=1.$$

Hence, dividing the first by xy, and the second by x^2y^2, we have

$$x^2+y^2=3$$
$$x^4+y^4=7$$

From double the last $\quad 2x^4+2y^4=1$

Subt. the square of the first $\quad x^4+2x^2y^2+y^4=9$

And we shall have $\quad x^4-2x^2y^2+y^4=5$

Or by extracting $\quad x^2-y^2=\sqrt{5}$

Also, as above, $\quad x^2+y^2=3$

Therefore by addition and subtraction

$$x^2=\frac{3}{2}+\frac{1}{2}\sqrt{5}$$

$$y^2=\frac{3}{2}-\frac{1}{2}\sqrt{5}$$

Whence, extracting these by the rule for binomial surds, we have $x=\frac{1}{2}(\sqrt{5}+1)$, and $y=\frac{1}{2}(\sqrt{5}-1)$.

Ex. 26. Here the equations are

$$\left.\begin{array}{l} x+y+z=23 \\ xy+xz+yz=167 \\ xyz=385 \end{array}\right\}$$

to find x, y, and z.

From what has been said in the Introduction, relative to the doctrine of equations, it is obvious that these numbers are the coefficients of a cubic equation which has its three roots equal to the several values of x, y, and z; whence we have at once

$$x^3-23x^2+167x-385=0.$$

The three roots of which equation, by the rules for cubics, are found to be 5, 7, and 11, the numbers sought.

Ex. 27. Here the given equations are

$$\begin{array}{l} xyz=231=a \\ xyw=420=b \\ xzw=660=c \\ yzw=1540=d \end{array}$$

Whence, multiplying these by each other, we have

$$x^3y^3w^3z^3=abcd=98611128000$$
$$\text{or } xywz=\sqrt[3]{abcd}=4620$$

Therefore, dividing this last equation by each of the given equations, we have

$$w=\frac{\sqrt[3]{abcd}}{a}=20$$

$$z=\frac{\sqrt[3]{abcd}}{b}=11$$

$$y=\frac{\sqrt[3]{abcd}}{c}=7$$

$$x=\frac{\sqrt[3]{abcd}}{d}=3$$

Ex. 28. Here the equations are

$$\begin{array}{l} x+yz=384=a \\ y+xz=237=b \\ z+xy=192=c \end{array}$$

From the first $x=a-yz$; which substituted in the second and third, gives

$$y+az-yz^2=b$$
$$z+ay-y^2z=c$$

Also from the first of these two equations

$$y=\frac{az-b}{z^2-1}.$$

Whence, substituting this value of y in the latter, we have

$$z+\frac{a^2z-ab}{z^2-1}-\frac{z(az-b)^2}{(z^2-1)^2}=c,$$

or $z(z^2-1)^2+a(az-b)(z^2-1)-z(az-b)^2=c(z^2-1)^2$

Which, by multiplying and involving the several factors, becomes

$$z^5-cz^4-2z^3+(2c+ab)z^2-(b^2+a^2-1)z=c-ab$$

Or in numbers

$$z^5-192z^4-2z^3+91392z^2-203624z=-90816.$$

An equation of the 5th degree, the integral root of which is $z=22$; whence $y=\frac{az-b}{z^2-1}=17$, and $x=a-yz=10$.

Ex. 29. Here we have the equations

$$x^2+xy=108=a$$
$$y^2+yz=\ \ 69=b$$
$$z^2+xz=580=c$$

Assume $x=my$, and $z=ny$, and these will become

$$y^2(m^2+m)=a$$
$$y^2(1\ +n)=b$$
$$y^2n(m+n)=c$$

Whence $y^2=\frac{a}{m^2+m}=\frac{b}{1+n}=\frac{c}{n(m+n)}$, or

$$a(1+n)=b(m^2+m)$$
$$an(m+n)=c(m^2+m)$$

From the first of these we have

$$n=\frac{b(m^2+m)-a}{a}$$

And this value of n, substituted in the second, gives

$$\frac{\{b(m^2+m)-a\}\times\{bm^2+(b+a)m-a\}}{a}=c(m^2+m)$$

And by reduction,

$$b^2m^4+(ab+2b^2)m^3+(b^2-ab-ac)m^2-a(a+2b+c)m+a^2=0;$$

Or in numbers

$$4761m^4+16974m^3-65331m^2-89208m=-11664,$$
$$\text{or}\quad 529m^4+\ 1886m^3-\ 7259m^2-\ 9912m=-1296.$$

From which is obtained $m=3$;

Consequently $n=\dfrac{b(m^2+m)-a}{a}=\dfrac{20}{3}, y=\sqrt{\left(\dfrac{a}{m^2+m}\right)}=3$, $x=my=9$, and $z=ny=20$, as required.

Ex. 30. Given $\left.\begin{aligned}x^2+xy+y^2&=5\\ x^4+x^2y^2+y^4&=11\end{aligned}\right\}$

Here, dividing the latter by the former, we have

$$x^2-xy+y^2=\frac{11}{5}$$

$$x^2+xy+y^2=5$$

And by addition $\quad x^2+y^2=\dfrac{18}{5}$

By subtraction $\quad xy=\dfrac{7}{5}$

Also, by adding and subtracting double the latter from the former, we have

$$x^2+2xy+y^2=\frac{18}{5}+\frac{14}{5}=\frac{32}{5}$$

$$x^2-2xy+y^2=\frac{4}{5}$$

Whence $x+y=\sqrt{\dfrac{32}{5}}=\dfrac{4}{5}\sqrt{10}$

And $x-y=\sqrt{\dfrac{4}{5}}=\dfrac{2}{5}\sqrt{5}$

Consequently $x=\dfrac{2}{5}\sqrt{10}+\dfrac{1}{5}\sqrt{5}$,

And $y=\dfrac{2}{5}\sqrt{10}-\dfrac{1}{5}\sqrt{5}$.

Which are the values of x and y as required.

Ex. 31. Here the given equation

$$x^{4n}-2x^{3n}+x^n=a$$

may be put under the form

$$(x^{2n}-x^n)^2-(x^{2n}-x^n)=a,$$

which being now a quadratic, we have

$$x^2n-x^n=\tfrac{1}{2}\pm\sqrt{(\tfrac{1}{4}+a)}$$

And this being also a quadratic, we derive from it

$$x^n=\tfrac{1}{2}\pm\sqrt{\{\tfrac{3}{4}\pm\sqrt{(\tfrac{1}{4}+a)}\}}$$

Therefore, by extraction,

$$x=\sqrt[n]{\{\tfrac{1}{2}\pm\sqrt{\left(\tfrac{3}{4}\pm\sqrt{(\tfrac{1}{4}+a)}\right)}\}}$$

Ex. 32. Let the people be represented by unity; then, if the first year's increase be denoted by $\frac{1}{n}$, the second year's increase will be $\frac{1}{n}+\frac{1}{n^2}$; hence $1+\frac{1}{n}$ will be the number of people at the end of the first year, and $1+\frac{2}{n}+\frac{1}{n^2}=\left(1+\frac{1}{n}\right)^2$ the number at the end of the second year. In like manner the number at the end of the third year will be $\left(1+\frac{1}{n}\right)^3$, and at the end of a century $\left(1+\frac{1}{n}\right)^{100}$, which, that the people may be doubled, must $=2$.

$$\therefore \log.\left(1+\frac{1}{n}\right)=\log. 2\div 100=\cdot 0030103=\log. 1\frac{1}{144}.$$

Whence $1+\frac{1}{n}=1\frac{1}{144}$; and consequently $\frac{1}{n}=\frac{1}{144}$, the answer.

Ex. 33. It is obvious that the least number of weights that can be used to weigh from 1*lb.* to 3*lb.* is two, viz., 1*lb.* and 3*lb.*, and if to these we add a 9*lb.*, we shall be able to weigh all the weights, 9 ± 1, 9 ± 2, 9 ± 3, 9 ± 4, viz., as far as 13*lb.*

Increasing again our weights by 3×9*lb.* $=27$*lb.* we shall be able to weigh 27 ± 1, 27 ± 2, 27 ± 3, &c., 27 ± 13, that is, to 40*lb.*; and in the same manner by the addition of three

times the last weight, viz. 81, we can weigh 81 ± 1, 81 ± 2, 81 ± 3, 81 ± 4, &c. 81 ± 40.

Therefore 1, 3, 9, 27, 81, are the weights required.*

Ex. 34. Let x=number of ducks, and y=number of geese.

Then by the question $30x+52y=28 \times 12=336$,

$$\text{Or, } x=\frac{336-52y}{30}=\frac{168-26y}{15}=11-y+\frac{3-11y}{15}.$$

But $\frac{3-11y}{15}$, or $\frac{6-22y}{15}=wh.$ also $\frac{6-22y}{15}+\frac{15y}{15}=\frac{6-7y}{15}$ $=wh.$ or $\frac{12-14y}{15}+\frac{15y}{15}=\frac{12+y}{15}=wh.=p.$

Whence $y=15p-12$; or taking $p=1$, we have $y=3$, and $x=6$.

Ex. 35. Let x be the number sought; then by the question $\frac{x-5}{6}$, $\frac{x-4}{5}$, $\frac{x-3}{4}$, $\frac{x-2}{3}$, and $\frac{x-1}{2}$ are to be all whole numbers.

Make $\frac{x-5}{6}=p$; then $x=6p+5$; substitute this in the 2nd, and we have

$$\frac{6p+1}{5}=p+\frac{p+1}{5}$$

$$\text{make } \frac{p+1}{5}=q, \text{ then } p=5q-1,$$

and consequently $x=30q-1$.

Substitute this in the 3d, and we have

$$\frac{30q-4}{4}=wh.; \text{ or } 7q-1+\frac{q}{2}=wh.$$

Whence $q=2r$; consequently $x=60r-1$.

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* The most general method of solving questions of this kind is by means of the ternary scale of notation.—See BARLOW's Theory of Numbers, chap. 10.

This, substituted in the 4th and 5th equations, gives whole numbers; therefore the general value of $x=60r-1$; if $r=1$, $x=59$, the least number sought.

Ex. 36. Let x be the year required; then if $x+9$, $x+1$, and $x+3$, be divided respectively by 28, 19, and 15, the remainders will be the cycles of the sun, the golden number, and the Roman indiction. Hence

$$\frac{x+9-18}{28},\ \frac{x+1-8}{19}\text{ and }\frac{x+3-10}{15}$$

$$\text{Or }\frac{x-9}{28},\ \frac{x-7}{19}\text{ and }\frac{x-7}{15},$$

must be all whole numbers.

$$\text{Let }\frac{x-9}{28}=p,\text{ then }x=28p+9,\text{ and }\frac{x-7}{19}=\frac{28p+2}{19}$$

$$=p+\frac{9p+2}{19}=wh.\text{ or }\frac{9p+2}{19}=wh.$$

$$\text{Let }\frac{9p+2}{19}=q;\text{ then }p=\frac{19q-2}{9}=2q+\frac{q-2}{9}=wh.,\text{ or}$$

$$\frac{q-2}{9}=wh.=r,\ \therefore\ q=9r+2$$

$$\therefore\ p=\frac{19q-2}{9}=19r+4,\text{ and }x=28p+9=532r+121.$$

Again, by substitution,

$$\frac{x-7}{15}=\frac{532r+114}{15}=35r+7+\frac{7r+9}{15}=wh.$$

$$\text{Or }\frac{7r+9}{15}=s;\text{ whence }r=\frac{15s-9}{7}=2s-1+\frac{s-2}{7}.$$

$$\text{Assume now }\frac{s-2}{7}=t;\text{ then }s=7t+2$$

Where s might be taken at pleasure; but as the least value is required, let $t=0$, then $s=2$, and $r=3$; and consequently $x=532r+121=1596+121=1717$, the year required.

Ex. 37. Given the equation $266x-87y=1$, to find the least values of x and y.

By transposition and division

$$y=\frac{266x-1}{87}=3x-\frac{5x+1}{87}=wh.$$

Let $\frac{5x+1}{87}=p$; then $x=\frac{87p-1}{5}=17p+\frac{2p-1}{5}$
$=wh.$; assume $\frac{2p-1}{5}=q$; then $p=\frac{5q+1}{2}=2q+\frac{q+1}{2}$
$=wh.$, or $\frac{q+1}{2}=wh.=r$; whence $q=2r-1$,

Where r may be taken at pleasure; if $r=1$, then $q=1$, and $p=3$; whence $x=\frac{87p-1}{5}=52$, and $y=153$, which are the least numbers that answer the conditions of the equation.

Ex. 38. Let x be one of the equal sides, y the base and z the perpendicular of the first triangle; and x', y' and z', the corresponding lines of the second triangle.

Then $2x+y$ is the perimeter of the first,
and $2x'+y'$ the perimeter of the second.

Also $\frac{yz}{2}$ the area of the first,

and $\frac{y'z'}{2}$ the area of the second.

Whence, by the question, we must have

$$2x+y=2x'+y'$$
$$\text{and} \quad yz=y'z'$$

$$\text{Also } x^2=\frac{y^2}{4}+z^2, \text{ and } x'^2=\frac{y'^2}{4}+z'^2$$

$$\text{Assume } x=r^2+s^2, \text{ and } \frac{y}{2}=r^2-s^2,$$

$$\text{also } x'=r^2+p^2, \text{ and } \frac{y'}{2}=r^2-p^2.$$

$$\text{Then } z=\left(x^2-\frac{y^2}{4}\right)^{\frac{1}{2}}=2rs,$$

$$\text{and } z'=\left(x'^2-\frac{y'^2}{4}\right)^{\frac{1}{2}}=2rp.$$

The perimeter of each triangle will be $4r^2$.

Also $\frac{yz}{2}=2rs(r^2-s^2)$, and $\frac{y'z'}{2}=2rp(r^2-p^2)$;

$$\text{whence } 2rs(r^2-s^2)=2rp(r^2-p^2)$$
$$\text{or } s(r^2-s^2)=p(r^2-p^2)$$

$$\text{And } r^2=\frac{s^3-p^3}{s-p}=s^2+sp+p^2$$

Therefore it remains to find

$$s^2+sp+p^2=\square.$$

Which latter condition has place, if we assume

$$s=m^2-n^2, \text{ and } p=n^2+2mn.$$

In which case $r=m^2+mn+n^2$

Whence

$$\begin{aligned}
x&=\ \ r^2+s^2\ \ =(m^2+mn+n^2)^2+(m^2-n^2)^2\\
y&=2(r^2-s^2)=2(m^2+mn+n^2)^2-2(m^2-n^2)^2\\
x'&=\ \ r^2+p^2\ \ =(m^2+mn+n^2)^2+(n^2+2mn)^2\\
y'&=2(r^2-p^2)=2(m^2+mn+n^2)^2-2(n^2+2mn)^2
\end{aligned}$$

Where m and n may be taken at pleasure.

If $m=2$ and $n=1$, then $x=58$, $y=80$, $x'=74$ and $y'=48$; or, since all these numbers are divisible by 2, we have

$$\left.\begin{matrix}x=29, & y=40\\ x'=37, & y'=24\end{matrix}\right\}\text{ as required.}$$

Ex. 39. Let x, y, and z represent the base, perpendicular, and hypothenuse, of the first triangle,

x', y', and z', those of the second, and
x'', y'', and z'', those of the third; then

$$\text{we have to find } \left.\begin{aligned}x^2+y^2&=z^2\\ x'^2+y'^2&=z'^2\\ x''^2+y''^2&=z''^2\end{aligned}\right\}\text{ all rational squares.}$$

$$\text{Also } xy=x'y'=x''y'';$$

And in order to fulfil the three first conditions,

$$\begin{aligned}
\text{Let } x&=m^2-n^2, \text{ and } y=2mn\\
x'&=m'^2-n'^2, \text{ and } y'=2m'n'\\
x''&=m''^2-n''^2, \text{ and } y''=2m''n''
\end{aligned}$$

Then it remains to find

$$(m^2-n^2)\times 2mn=(m'^2-n'^2)\times 2m'n'$$
$$(m'^2-n'^2)\times 2m'n'=(m''^2-n''^2)\times 2m''n''$$

Which equations may be resolved into the factors

$$(m+n)(m-n)\,mn=(m'+n')(m'-n')m'n'$$
$$(m''+n'')(m''-n'')m''n''=(m'+n')(m'-n')m'n'.$$

It is only necessary so to equate the factors of each of these equations, that the reduction of them may be in the same ratio to two of the quantities.

For which purpose, let $m+n=2n'$

and $m=m'$

then $m-n=2(m'-n')$

and $n=2n'-m'$.

But since the product of the preceding factors are equal, we must have

$$2n'-m'=\frac{m'+n'}{4},\text{ or }7'n=5m'$$

Again, in the second equation we may assume

$$m''+n''=3n'$$
$$n''=m'$$

and $m''-n''=\frac{1}{2}(m'-n')$

then $m''=3n'-m'$.

Where again, because the product of the factors are equal, we must have

$$\tfrac{3}{2}(3n'-m')=m'+n',\text{ or }5m'=7n',$$

which is the same ratio as before.

Assuming, therefore, $m'=7$, and $n'=5$, we have $m=7$ and $n=3$; $m''=8$ and $n''=7$;

Whence $x=m^2-n^2=40$; $y=2mn=42$

$x'=m'^2-n'^2=24$; $y'=2m'n'=70$

$x''=m''^2-n''^2=15$; $y''=2m''n''=112$.

Ex. 40. Given $x^{\frac{1}{x}}=1\cdot2655$ to find x.

Here a few trials show that x is between $1\cdot3$ and $1\cdot4$.

Where, if $x=1\cdot3$, then $\dfrac{\log.\ 1\cdot3}{1\cdot3}=\cdot087649$

$\log.\ 1\cdot2655=\cdot102262$

Error $-\cdot014613$

$$\text{And if } x=1\cdot4, \text{ then } \frac{\log.\ 1\cdot4}{1\cdot4}=\cdot104377$$

$$\log.\ 1\cdot2655=\cdot102262$$

$$\text{Error} \quad +\cdot002115$$

Hence $\cdot016728 : \cdot1 :: \cdot014613 : .08736$

Therefore $1\cdot3+\cdot08736=1\cdot38736$. Ans.

Ex. 41. By a few trials we learn that x is between 4 and 5, and y between 5 and 6.

Assume $x=4\cdot7$, and the second equation becomes $y^{4\cdot7}=3000$; therefore $y=3000^{\frac{1}{4\cdot7}}$, and $\log.\ y=\frac{1}{4\cdot7}\times\log.\ 3000=\frac{1}{4\cdot7}\times3\cdot4771213=0\cdot7398130=\log.5\cdot493$; hence $y=5\cdot493$. Therefore $\log.\ x^y=5\cdot493\times\log.\ 4\cdot7=5\cdot493\times0\cdot6720979=3\cdot6918338$, the first result, which is in defect; for by the question $\log.\ x^y=\log.\ 5000=3\cdot6989700$. It will be found by increasing x that the result would be still greater in defect; x is therefore less than $4\cdot7$, and assuming $x=4\cdot69$, the first equation will give $\log.\ y=\frac{1}{4\cdot69}\times\log.\ 3000=0\cdot7413905=\log.\ 5\cdot513$, or $y=5\cdot513$. Hence $\log.\ x^y=5\cdot513\times\log.\ 4\cdot69=5\cdot513\times0\cdot6711728=3\cdot7001756$, the second result, which is in excess. Wherefore

3·7001756		4·69		3·7001756		
3·6918338		4·70		3·6989700		
83418	:	0·01	::	12056	:	0·001445;

consequently $x=4\cdot691445$, and $\log.\ y=\frac{1}{4\cdot691445}\times\log.\ 3000=0\cdot7411621=\log.\ 5\cdot510133$, and $\therefore y=5\cdot510133$.

By a similar process we may find $y=5\cdot51$ nearly; and repeating the operations again on x and y, we have

$$x=4\cdot691445 \text{ and } y=5\cdot510132.$$

Ex. 42. By a few trials we learn that x is nearly $=4$, and y nearly $=3$.

Assume $x=4\cdot01$; then $x^x=4\cdot01^{4\cdot01}$, or log. $x^x=4.01\times$ log. $4\cdot01=2\cdot418609$; hence $x^x=262\cdot186$, and $y^y=285-x^x=22\cdot814$;

$$\therefore y\times \text{log. } y= \text{log. } 22\cdot814=1\cdot358201.$$

By trial y is found to be between $2\cdot91$ and $2\cdot92$; hence substituting each of these for y, we have
$2\cdot91\times$ log. $2\cdot91=1\cdot349929$, and $2\cdot92\times$ log. $2\cdot92=1\cdot358918$. Therefore

1·358918	2·92	1·358918
1·349929	2·91	1·358201
8989 :	0·01 ::	717 : 0·0008

whence $y=2\cdot92-0\cdot0008=2\cdot9192$.

We have now to see how the values $x=4\cdot01$ and $y=2\cdot9192$, will satisfy the equation $y^x-x^y=14$:

$$\text{log. } y^x=x\times \text{log. } y=1\cdot865708,\ \therefore y^x=73\cdot4020,$$
$$\text{log. } x^y=y\times \text{log. } x=1.760699,\ \therefore x^y=57\cdot6367.$$

Whence $y^x-x^y=15\cdot7653$, the first result, which should have been 14, and is therefore in excess. Were we to take x less than $4\cdot01$, the result would be still more in excess. Assume, therefore, $x=4\cdot02$; then $x^x=4\cdot02^{4.02}$, and log. $x^x=2\cdot428989$; $\therefore$ $x^x=268\cdot528$, and $y^y=285-x^x=16\cdot472$;

$$\therefore y\times \text{log. } y= \text{log. } 16\cdot472=1\cdot216746.$$

By trial y is found to be between $2\cdot75$ and $2\cdot76$.

Whence $2\cdot75\times$ log. $2\cdot75=1.208165$, and $2\cdot76\times$ log. $2\cdot76=1\cdot216909$. Therefore

1·216909	2·76	1·216909
1·208165	2·75	1·216746
8744 :	0·01 ::	163 : 0·0002

and $y=2\cdot76-0\cdot0002=2\cdot7598$;

$$\therefore \text{log. } y^x=4\cdot02 \times \text{log. } 2\cdot7598=1\cdot772328$$
$$\text{log. } x^y=2\cdot7598\times \text{log. } 4\cdot02 =1\cdot667543$$

Hence $y^x=59\cdot2009$, and $x^y=46\cdot5096$; $\therefore y^x-x^y=12\cdot6913$, the second result, which should have been 14 and is consequently in defect. Wherefore

15·7653	4·01	14
12·6913	4·02	12·6913
3·0740 :	0·01 ::	1·3087 : 0·004

$$\therefore x=4\cdot02-0\cdot004=4\cdot016$$

Again, we will now commence a recomputation with this value of x, so that $x^x=4{\cdot}016^{4{\cdot}016}=265{\cdot}9717$, and $y^y=285-x^x=19{\cdot}0283$;

$$\therefore y\times \log. y= \log. 19{\cdot}0283=1{\cdot}2794000.$$

We learn by trial that y is between $2{\cdot}83$ and $2{\cdot}84$, so that $2{\cdot}83\times \log. 2{\cdot}83=1{\cdot}2785555$, and $2{\cdot}84\times \log. 2{\cdot}84=1{\cdot}2874240$. Whence

1·2874240	2·84	1·2794000
1·2785555	2·83	1·2785555

$$88685 : 0{\cdot}01 :: \quad 8445 : 0{\cdot}000952$$

$$\therefore y=2{\cdot}83+0{\cdot}000952=2{\cdot}830952.$$

$$\therefore \log. y^x=4{\cdot}016\times \log. 2{\cdot}830952=1{\cdot}8149609$$

$$\log. x^y=2{\cdot}830952\times \log. 4{\cdot}016=1{\cdot}7093110$$

Hence $y^x-x^y = 65{\cdot}30717 - 51{\cdot}20484 = 14{\cdot}10233$, the first result, which is in excess. Assume therefore $x=4{\cdot}017$, and we shall have $x^x=4{\cdot}017^{4{\cdot}017}=266{\cdot}6082$; hence $y^y=285-x^x=18{\cdot}3918$, and therefore

$$y\times \log. y= \log. 18{\cdot}3918=1{\cdot}2646243.$$

By trial we learn that y is between $2{\cdot}81$ and $2{\cdot}82$; now $2{\cdot}81\times \log. 2{\cdot}81=1{\cdot}2608647$, and $2{\cdot}82\times \log. 2{\cdot}82=1{\cdot}2697025$. Therefore

1·2697025	2·82	1·2646243
1·2608647	2·81	1·2608647

$$88378 : 0{\cdot}01 :: \quad 37596 : 0{\cdot}004254$$

$$\text{and } y=2{\cdot}81+0{\cdot}004254=2{\cdot}814254$$

$$\therefore \log. y^x=4{\cdot}017\times \log. 2{\cdot}814254=1{\cdot}8050924$$

$$\log. x^y=2{\cdot}814254\times \log. 4{\cdot}017=1{\cdot}6995331$$

Whence $y^x-x^y=63{\cdot}83993-50{\cdot}06487=13{\cdot}77506$, the second result, which is in defect. Wherefore

14·10233	4·016	14·10233
13·77506	4·017	14·00000

$$32727 : 0{\cdot}001 :: \quad .10233 : 0{\cdot}000313$$

$$\text{and } x=4{\cdot}016+0{\cdot}000313=4{\cdot}016313.$$

Hence $x^x=4{\cdot}016313^{4{\cdot}016313}=266{\cdot}1707$, and $y^y=285-x^x=18{\cdot}8293$;

$$\therefore y\times \log. y= \log. 18{\cdot}8293=1{\cdot}2748342.$$

We find by trial that y is between $2{\cdot}82$ and $2{\cdot}83$; now $2{\cdot}82\times \log. 2{\cdot}82=1{\cdot}2697025$, and $2{\cdot}83\times \log. 2{\cdot}83=1{\cdot}2785555$. Therefore

$$\begin{array}{ccc} 1{\cdot}2785555 & 2{\cdot}83 & 1{\cdot}2785555 \\ 1{\cdot}2697025 & 2{\cdot}82 & 1{\cdot}2748342 \\ \hline 88530 : & 0{\cdot}01 :: & 37213 : 0{\cdot}004203 \end{array}$$

$$\text{and } y = 2{\cdot}83 - 0{\cdot}004203 = 2{\cdot}825793$$

We conclude with $x = 4{\cdot}016313$, and $y = 2{\cdot}825793$, which on trial will be found very near the truth.

Ex. 43. Here, calling $2x$ the number sought, we have to find

$$2x + 1 = \square, \text{ and } x + 1 = \square.$$

Assume $x = m^2 - 2m$; then $x + 1 = (m-1)^2$ a square as required; and therefore it remains to find $2m^2 - 4m + 1 =$ a square.

$$\text{Assume } 1 - 4m + 2m^2 = (rm - 1)^2,$$

$$\text{then } 2m^2 - 4m = r^2m^2 - 2rm,$$

$$\text{or } m = \frac{2r - 4}{r^2 - 2};$$

$$\text{Since } r \text{ may be a fraction, let } r = \frac{p}{q},$$

Then $m = \dfrac{2pq - 4q^2}{p^2 - 2q^2}$; where p and q may be taken at pleasure, provided the result be integral.

If we take $p = 4$, and $q = 3$, then $m = 6$ and $2x = 48$
$p = 7$, and $q = 5$, then $m = 30$ and $2x = 1680$,
numbers answering the required conditions; and various others might be found by giving different values to p and q.

Ex. 44. Here x and y being taken to denote the numbers sought, we have to find

$$x^2 + y^2 = \square$$
$$x^3 + y^3 = \square.$$

Assume $x^2 + y^2 = (ry - x)^2$; then we have

$$x^2 + y^2 = r^2y^2 - 2ryx + x^2, \text{ or}$$

$$y = r^2y - 2rx, \text{ or}$$

$$x = \frac{(r^2 - 1)y}{2r}.$$

$$\text{Whence } x^3 + y^3 = \frac{(r^2 - 1)^3y^3}{8r^3} + y^3$$

which is to be a square, or

$$\frac{(r^2-1)^3+8r^3}{2r}y=\square=s^2,$$

and consequently $y=\dfrac{2rs^2}{(r^2-1)^3+8r^3}$,

Where r and s may be taken at pleasure.

If $r=2$, then $y=\dfrac{4s^2}{91}$ and $x=\dfrac{3s^2}{91}$; so that taking $s=91$,

we have $y=364$ and $x=273$,

which are the numbers required.

Ex. 45. Let x^2 and y^2 be the numbers sought; then the two latter conditions will be fulfilled, and it will only remain to find

$$x^2+y^2=\text{ a cube.}$$

For which purpose let $x=rz$, and $y=sz$; then

$$r^2z^2+s^2z^2=\text{ a cube.}$$

Assume $r^2z^2+s^2z^2=\dfrac{z^3}{v^3}$;

$$\text{Then } z=v^3(r^2+s^2)$$
$$x=rv^3(r^2+s^2)$$
$$y=sv^3(r^2+s^2),$$

Where r, s, and v may be assumed at pleasure. If $r=2$, $s=1$, and $v=1$, then $x=10$ and $y=5$; consequently $x^2=100$, and $y^2=25$. If $r=3$, $s=1$, and $v=1$, then $x=30$, and $y=10$; therefore $x^2=900$, and $y^2=100$.

And, by giving different values to r, s, and v, an indefinite number of other answers may be found.

Ex. 46. This is the same question, except a little variation in the enunciation, as Ex. 19 of the Diophantine Problems, where we found generally

$$x=(p^2-q^2),\quad y=2pq,\qquad z=2pq\frac{(p^2-q^2)}{(p^2+q^2)}.$$

Hence, if each of these be multiplied by (p^2+q^2),

$$p^4-q^4,\ 2pq(p^2+q^2) \text{ and } 2pq(p^2-q^2)$$

will be the integral roots in the present question; p and q being any unequal numbers whatever.

If $p=2$, and $q=1$; then 15^4, 20^4, and 12^4, are the biquadrates required.

Ex. 47. Let ax^2, ay^2 and $\frac{ay^4}{x^2}$, represent the three numbers in geometrical progression; then by the question

$$(y^2-x^2\ )a=\square$$

$$\left(\frac{y^4-x^4}{x^2}\right)a=\square$$

$$\left(\frac{y^4-x^2y^2}{x^2}\right)a=\frac{y^2}{x^2}(y^2-x^2)a=\square\,.$$

Hence making $y^2-x^2=am^2$, and $y^4-x^4=an^2$, all the conditions will be satisfied.

$$\text{Whence } y^2+x^2=\frac{n^2}{m^2}.$$

Assume, therefore, $x=p^2-q^2$, and $y=2pq$; where p and q may be taken at pleasure. If $p=2$, and $q=1$, then $x=3$, and $y=4$; and the numbers are

$$9a,\quad 16a, \text{ and } \frac{256a}{9}, \text{ or}$$

$$81a,\ 144a, \text{ and } 256a,$$

a being $=\frac{y^2-x^2}{m^2}$; where m^2 may be any square factor whatever of y^2-x^2.

In the present instance let $m=1$, then $a=7$, and the required numbers are 567, 1008 and 1792.

Ex. 48. Let x, y, and z denote the three numbers, and assume $x+y=a^2$, $x+z=b^2$, $y+z=c^2$; then by subtraction

$$\left.\begin{array}{l} x-z=a^2-c^2 \\ x-y=b^2-c^2 \\ y-z=a^2-b^2 \end{array}\right\} \text{all squares.}$$

We have, therefore, only to find such values of a, b, and c, as will satisfy the latter conditions; for in that case the former must have place.

But such values of a, b, and c, have been found in Ex. 20, Diophantine Problems, viz. $697^2=485809$, $185^2=34225$, and $153^2=23409$.

Hence, considering these quantities as known, we have by the common rules

$$x=\frac{a^2+b^2-c^2}{2}=248312\tfrac{1}{2}$$

$$y=\frac{a^2+c^2-b^2}{2}=237496\tfrac{1}{2}$$

$$z=\frac{-a^2+b^2+c^2}{2}=-214087\tfrac{1}{2}.$$

Since the sum and diff. of every two are to be squares, it is evident that we may regard z as positive; also multiplying each by 4, in order to avoid fractions, we have $x=993250$, $y=949986$, and $z=856350$.

Which numbers answer the conditions of the question; and various others may be had by finding different values for a, b, and c.

Ex. 49. Let x and y be the numbers sought; then we have to find

$$x+y=\textit{a square},$$
$$x^2+y^2=\textit{a biquadrate}.$$

First, in order to make $x^2+y^2=$ a square, assume

$$x=p^2-q^2, \text{ and } y=2pq;$$

Then shall $\qquad x^2+y^2=(p^2+q^2)^2=\square.$

But when $x^2+y^2=$ a biquadrate, p^2+q^2 must be a square; assume therefore, again,

$$p=r^2-s^2, \text{ and } q=2rs,$$

Then we shall have

$$x^2+y^2=(p^2+q^2)^2=(r^2+s^2)^4, \text{ a biquadrate, as required.}$$

And it now only remains to find

$$x+y=r^4+4r^3s-6r^2s^2-4rs^3+s^4=\square.$$

Hence, in order to reduce this to a more convenient form for solution, substitute $r=\frac{3s}{2}+t$, and the above formula, after multiplying by 16, reduces to

$$s^4+296s^3t+408s^2t^2+160st^3+16t^4=\square,$$

which assume

$$=(s^2+148st-4t^2)^2=s^4+296s^3t+21896s^2t^2-1184st^3+16t^4$$

By cancelling the like terms in both, this reduces to

$$21896s - 1184t = 408s + 160t.$$

$$\text{Whence } \frac{s}{t} = \frac{1344}{21488} = \frac{84}{1343}.$$

Assume, therefore, $s = 84$, and $t = 1343$, and we shall have $r = \frac{3s}{2} + t = 1469$, and consequently

$$x = r^4 - 6r^2s^2 + s^4 = 4565486027761$$
$$y = 4r^3s - 4rs^3 = 1061652293520$$

for the numbers sought.

Ex. 50. The solution of this question is intimately connected with that of Ex. 48; for if we here call w, x, y, and z, the four numbers, and at the same time make $w = x + y + z$, we shall have to find

$$w - x = y + z = \square$$
$$w - y = x + z = \square$$
$$w - z = x + y = \square$$

also $x - y = \square$, $x - z = \square$, and $y - z = \square$.

It is, therefore, only necessary that x, y, and z, may be such numbers that the sum and difference of every two of them may be a square, which are the conditions of Ex. 48, where we found the three numbers to be

$$x = 993250$$
$$y = 949986$$
$$z = 856350$$

and consequently $w = 2799586$,

which numbers answer the conditions of the question.

And if, in our 48th Example, we had taken $a = 2165$, $b = 2067$, and $c = 2040$, we should have found

$$x = 2399057$$
$$y = 2288168$$
$$z = 1873432$$

and $w = 6560657$,

which are the numbers given in the answer in the Introduction.

Ex. 51. Here the proposed series may be put under the form

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}, \&c. =\frac{3}{2}\times\frac{1}{3}$$
$$+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}, \&c. =\frac{3}{2}\times\frac{1}{9}$$
$$+\frac{1}{27}+\frac{1}{81}, \&c. =\frac{3}{2}\times\frac{1}{27}$$
$$+\frac{1}{81}, \&c. =\frac{3}{2}\times\frac{1}{81},$$

Whence the whole sum is

$$=\frac{3}{2}\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\&c.\right)=\frac{3}{2}\times\frac{3}{2}\times\frac{1}{3}=\frac{3}{4}.$$

Ex. 52. This may be separated into the two series

$$\frac{3}{4}+\frac{27}{64}+\frac{243}{1024}+\&c.=\frac{16}{7}\times\frac{3}{4}=\frac{12}{7}$$
$$\frac{9}{16}+\frac{81}{256}+\frac{729}{4096}+\&c. =\frac{16}{7}\times\frac{9}{16}=\frac{9}{7},$$

Whence the whole series $=\frac{12}{7}-\frac{9}{7}=\frac{3}{7}$. Answer.

Ex. 53. The methods laid down in the Introduction are not at all applicable to the present inquiry. The following solution, depending on the doctrine of equations, is as elementary as the question admits of.

By Bonnycastle's Trigonometry, art. 50,

$$\sin z=z(1-\frac{z^2}{2.3}+\frac{z^4}{2.3.4.5}-\frac{z^6}{2.3.4.5.6.7}, \&c.)$$

whose factors are $(z\pm\pi)$, $(z\pm2\pi)$, $(z\pm3\pi)$, &c., because $\pm\pi$, $\pm2\pi$, $\pm3\pi$, &c. being substituted for z, the equation $\sin z=0$, will always be satisfied, (π being the semicircumference of a circle); and z being a common factor, we shall have one root $z=0$. Hence

$$(z+\pi).(z-\pi).(z+2\pi).(z-2\pi).(z+3\pi)(z-3\pi), \&c.$$

or its equal

$$(z^2-\pi^2).(z^2-4\pi^2).(z^2-9\pi^2), \&c.$$
$$=1-\frac{z^2}{2.3}+\frac{z^4}{2.3.4.5}-\frac{z^6}{2.3.4.5.6.7}$$

Now, by the theory of equations the last coefficient but one, divided by the last coefficient, is equal to the sum of the reciprocals of the roots

$$\therefore \ \frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}, \text{ \&c.} \qquad =\frac{1}{2.3}$$

$$\text{or } 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}, \text{ \&c.} =\frac{\pi^2}{6}$$

$$\text{Hence} \qquad \frac{2}{2^2} \qquad +\frac{2}{4^2}, \text{ \&c.}=\frac{\pi^2}{12}$$

$$\therefore \ 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}, \text{ \&c.} =\frac{\pi^2}{12}= \cdot 822467$$

the sum required.

Ex. 54. Here, by the rules for arithmetical progression, the nth term $=5+(n-1)$.

Therefore $\{10+(n-1)\}\frac{n}{2}=\frac{n}{2}(n+9)$ is the sum required.

Ex. 55. The 25th term of the progression

1, 2, 4, 8, 16, &c.

$=2^{24}=16777216$; therefore the 25th term of the proposed series is $2^{16777216}$; that is, the 16777216th power of 2.

$$\begin{array}{lr} \text{Now the log. of } 2= & 0.3010300 \\ \text{Mult. by} & 16777216 \\ \hline \text{Gives} & 5050445\cdot3324800 \end{array}$$

for the logarithm of the 25th term; consequently the index being 5050445, the number of integers will be 5050446.

Ex. 56. This series is the same as

$$(2^2-2)+(4^2-4)+(6^2-6)+, \text{ \&c. viz.}$$

$$\begin{cases} 4(1^2+2^2+3^2+4^2+, \text{ \&c. } 100^2)- \\ 2(1+2+3+4+, \text{ \&c. } 100). \end{cases}$$

But the sum of the former

$$=4\times\frac{n(n+1)(2n+1)}{6}=\frac{4\times100\times101\times201}{6}$$

$$=1353400,$$

And the sum of the latter $=2\times\frac{101\times100}{2}=10100,$

Whence

$1353400 - 10100 = 1343300$, the sum required.

Ex. 57. This series is the same as Ex. 2, page 139.

Ex. 58. By the differential formula we have

$$\begin{array}{llll} 35, & 72, & 111, & 152, \quad \&c. \\ & 37, & 39, & 41, \quad \&c. \text{ 1st diff.} \\ & & 2, & 2, \quad \&c. \text{ 2d diff.} \end{array}$$

Whence

$$a=35,\ d'=37,\ d''=2,\ n=25;$$

$$na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''=$$

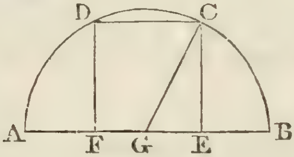
$$(35\times 25)+(12\times 25\times 37)+(25\times 8\times 23)=875+11100+$$
$$4600=16575, \text{ the sum sought.}$$

APPLICATION OF ALGEBRA TO GEOMETRY.

MISCELLANEOUS PROBLEMS.

PROBLEM I.

Let ABCD be the given semicircle; AB, its diameter; G, its centre; and CDFE, the required square.

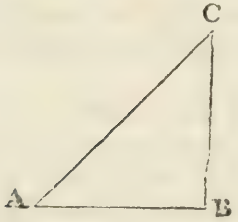


Then, since $DF=CE$, we have $FG=GE$.

Let therefore $AB=d$, or $CG=\frac{1}{2}d$; also $CE=x$, and consequently $GE=\frac{1}{2}x$; then by (Euc. I. 47)

$$GE^2+CE^2=CG^2, \text{ or } \tfrac{1}{4}x^2+x^2=\tfrac{1}{4}d^2$$
$$\text{Whence } 5x^2=d^2, \text{ or } x=d\sqrt{\tfrac{1}{5}}=\tfrac{1}{5}d\sqrt{5}.$$

2. Let ABC be the required right angled triangle; in which take $AC=13=h$, $AB=x$, $BC=y$, and the given difference $AB \backsim BC=7=d$.



Then by Euc. I. 47, we shall have

$$x^2+y^2=h^2$$
$$x-y=d.$$

Squaring the second equation, and subtracting it from double the first, we have

$$x^2+2xy+y^2=2h^2-d^2,$$

Consequently $x+y=\sqrt{(2h^2-d^2)}$,

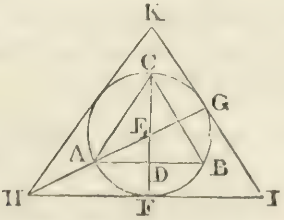
but $x-y=d$,

Whence $x=\frac{1}{2}\sqrt{(2h^2-d^2)}+\frac{1}{2}d=12$,

and $y=\frac{1}{2}\sqrt{(2h^2-d^2)}-\frac{1}{2}d=\ 5$.

Ex. 3. Let AFGC be the given circle, and ABC the required inscribed equilateral triangle.

Join A and the centre E; also join CE, and produce it to D.



Then by Euc. (III. 3) the angle D is a right angle, and the triangles ADE and ADC are similar.

But AD$=\frac{1}{2}$AC; therefore also DE$=\frac{1}{2}$AE.

Let then AE$=$ radius $=\frac{1}{2}d$; and consequently ED$=\frac{1}{2}$AE$=\frac{1}{4}d$; also put AB$=x$, or AD$=\frac{1}{2}x$.

Then by Euc. (I. 47) $\frac{1}{4}x^2=\frac{1}{4}d^2-\frac{1}{16}d^2=\frac{3}{16}d^2$;

Whence $x=\sqrt{\frac{3}{4}d^2}=\frac{1}{2}d\sqrt{3}$, the side of the inscribed triangle.

Again, produce CD to F, and AE both ways to G and H; and draw HI, IK, perpendicular to EF and EG.

Then it is obvious that HKI will be the circumscribing equilateral triangle, and, as before, that $EF=\frac{1}{2}HE$, and $HF=\frac{1}{2}HI$.

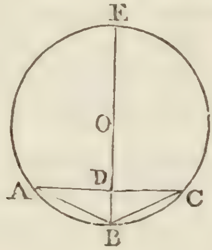
Let $HI=y$, and we shall have

$HE^2=EF^2+HF^2$; whence since $EF=\frac{1}{2}d$, $HE=d$, we have

$$d^2=\tfrac{1}{4}d^2+\tfrac{1}{4}y^2, \text{ or } y^2=3d^2, \text{ or } y=d\sqrt{3},$$

$=$ the side of the circumscribing triangle.

Ex. 4. The straight line joining the extremities of two adjacent sides of a regular decagon inscribed in a circle, is obviously the side of a regular pentagon inscribed in the same circle. Now, it appears, from Euc. IV. 10, that the side of a regular decagon inscribed in a circle is found by dividing the radius of the circle into extreme and mean ratio, the greater part of which is the side of the decagon.



Hence calling the diameter $EB=2r$, or radius $OB=r$, and the side BC of the decagon $=x=OD$ the greater extreme, we have $r(r-x)=x^2$, or $x^2+rx=r^2$.

$$\text{Whence } BC=x=-\tfrac{1}{2}r+\tfrac{1}{2}\sqrt{5r^2}=\frac{r}{2}(-1+\sqrt{5}).$$

Again by Euc. VI. 8. cor. $EB : BC :: BC : BD=\frac{r}{4}(3-\sqrt{5})$

Whence $DC^2=BC^2-BD^2=\frac{r^2}{4}(6-2\sqrt{5})-\frac{r^2}{16}(14-6\sqrt{5})$

$=\frac{r^2}{16}(24-8\sqrt{5})-\frac{r^2}{16}(14-6\sqrt{5})=\frac{r^2}{16}(10-2\sqrt{5})$, or $DC=$

$\frac{r}{4}\sqrt{(10-2\sqrt{5})}$, and therefore $AC=2DC=\frac{r}{2}\sqrt{(10-2\sqrt{5})}=$

$\frac{d}{4}\sqrt{(10-2\sqrt{5})}$ the side of the pentagon.

Ex. 5. Let $x=$ the length of the rectangle, and

$y=$ the breadth; then

$2x+2y=$perimeter, and $xy=$area.

Now since the side of the square $=a$, its perimeter $=4a$, and its area $=a^2$, we shall have

$$\left.\begin{array}{r} xy=\tfrac{1}{2}a^2 \\ 2x+2y=4a \end{array}\right\} \text{ or } \left.\begin{array}{r} xy=\tfrac{1}{2}a^2 \\ x+\ y=2a \end{array}\right\}$$

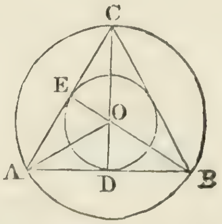
$$\begin{array}{ll} \text{Whence} & x^2+2xy+y^2=4a^2 \\ \text{Subtract} & \quad 4xy \quad\ =2a^2 \\ \text{We have} & x^2-2xy+y^2=2a^2, \text{ or} \\ & x-y=a\sqrt{2} \\ \text{But} & x+y=2a \end{array}$$

Whence by addition and subtraction

$$x=a+\tfrac{1}{2}a\sqrt{2}=a(1+\tfrac{1}{2}\sqrt{2})$$
$$y=a-\tfrac{1}{2}a\sqrt{2}=a(1-\tfrac{1}{2}\sqrt{2})$$

the length and breadth as required.

Ex. 6. Let ABC be the given equilateral triangle, and bisect the two sides AB, AC, by the two perpendiculars DO, EO;



Then shall the point O be the centre of the circumscribing circle; and OA its radius.

Also, AO, OB, manifestly bisect the angles A and B; O is therefore the centre of the inscribed circle, and OD its radius, Euc. (IV. 4) and (IV. 5.)

Again, if DO be produced, it will pass through C, and bisect the angle ACB.

Consequently the triangles ACD and AOD are similar; and since AD$=\frac{1}{2}$AC, therefore DO$=\frac{1}{2}$AO.

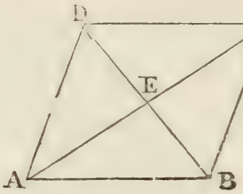
Let now AB$=s$ the given side, or AD$=\frac{1}{2}s$; also DO$=x$, and consequently AO$=2x$; then AO$^2=$AD$^2+$DO^2, or

$$4x^2=\tfrac{1}{4}s^2+x^2; \text{ whence}$$

$$3x^2=\tfrac{1}{4}s^2, \text{ or } x=\tfrac{1}{2}s\sqrt{\tfrac{1}{3}}=\tfrac{5}{3}\sqrt{3}=2\cdot 88675$$
$$\text{and } 2x=s\ \sqrt{\tfrac{1}{3}}=\tfrac{10}{3}\sqrt{3}=5\cdot 77350$$

the two radii required.

Ex. 7. Let ABCD be the rhombus, and AC, BD, its two diagonals, intersecting each other in E.



Also let the perimeter $=4p$, that is, each side of the rhombus $=p$, and the sum of the two diagonals $=s$.

Then since the diagonals of parallelograms bisect each other, CE+EB$=\frac{1}{2}s$.

And because the three sides of the triangle DEC, and CEB, are respectively equal to each other, the angles DEC and BEC are equal, and therefore the angles at E are each equal to a right angle.

Therefore calling AC$=x$ and DB$=y$, or CE$=\frac{1}{2}x$ and EB$=\frac{1}{2}y$; we have

$$\left.\begin{aligned} x+y&=s \\ \tfrac{1}{4}x^2+\tfrac{1}{4}y^2&=p^2 \end{aligned}\right\}$$

From 8 times the latter

$$2x^2+2y^2=8p^2$$

Subtract $(x+y)^2=x^2+2xy+y^2=s^2$

And we have $x^2-2xy+y^2=8p^2-s^2$, or

$$x-y=\sqrt{(8p^2-s^2)}$$

But $x+y=s$;

Whence by addition and subtraction,

$$x=\tfrac{1}{2}s+\tfrac{1}{2}\sqrt{(8p^2-s^2)}$$
$$y=\tfrac{1}{2}s-\tfrac{1}{2}\sqrt{(8p^2-s^2)}$$

Or since $s=8$, and $p=3$; we have

$$x=4+\tfrac{1}{2}\sqrt{8}=4+\sqrt{2}$$
$$y=4-\tfrac{1}{2}\sqrt{8}=4-\sqrt{2}$$

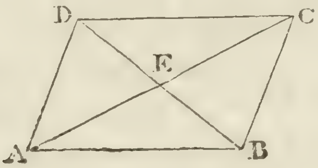
Which are the two diagonals required.

Ex. 8. Here the three sides of the right angled triangle are x^{3x}, x^{2x}, x^x; and since the square of the longest side is equal to the sum of the squares of the other two (Euc. I. 47), we shall have

$$x^{6x}=x^{4x}+x^{2x};\text{ or}$$
$$x^{4x}-x^{2x}=1$$

$$\text{Whence } x^{2x}=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}=1\cdot618034$$
$$\text{And } \quad x^{x}=\sqrt{1\cdot618034}=1\cdot272020\,;$$
$$\text{Therefore } \frac{x^{2x}\times x^{x}}{2}=1\cdot029086 \text{ the area.}$$

Ex. 9. It is a well-known geometrical theorem, that the diagonals of a parallelogram bisect each other; and that the sum of their squares is equal to the sum of the squares of the four sides of the parallelogram.



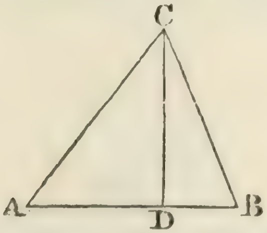
If therefore we represent the given parallelogram by the figure ABCD, and make its side BC$=a$, DC$=b$, DB$=d$, and AC$=x$, we have from the above theorem

$$x^2+d^2=2a^2+2b^2, \text{ or}$$
$$x^2=2a^2+2b^2-d^2, \text{ or}$$
$$x=\sqrt{(2a^2+2b^2-d^2)}$$

Which is the diagonal required.

Ex. 10. Let ABC be the proposed triangle, CD its perpendicular, and AD, DB the segments of the base.

Put CD$=300=p$, AC$+$BC$=1155=s$, and AD$-$DB$=495=d$.



Also put AD$=y+\frac{1}{2}d$, and DB$=y-\frac{1}{2}d$; then (Euc. I. 47.)

$$\sqrt{\{(y+\tfrac{1}{2}d)^2+p^2\}}=\text{AC}$$
$$\sqrt{\{(y-\tfrac{1}{2}d)^2+p^2\}}=\text{CB}\,;$$

And by the question

$$\sqrt{\{(y+\tfrac{1}{2}d)^2+p^2\}}+\sqrt{\{(y-\tfrac{1}{2}d)^2+p^2\}}=s.$$

Squaring both sides, and transposing

$$2\sqrt{\{(y+\tfrac{1}{2}d)^2+p^2\}}\times\sqrt{\{(y-\tfrac{1}{2}d)^2+p^2\}}=$$
$$s^2-2y^2-\tfrac{1}{2}d^2-2p^2;$$

or, in order to simplify the expression, let

$$s^2-\tfrac{1}{2}d^2-2p^2=2m; \text{ then}$$
$$\sqrt{\{(y+\tfrac{1}{2}d)^2+p^2\}}\times\sqrt{\{(y-\tfrac{1}{2}d)^2+p^2\}}=m-y^2.$$

Squaring again both sides, and actually performing the multiplication of the first two factors, we have

$$(y^2-\tfrac{1}{4}d^2)^2+2p^2(y^2+\tfrac{1}{4}d^2)+p^4=m^2-2my^2+y^4;$$

and by involving and collecting the terms

$$\tfrac{1}{16}d^4-\tfrac{1}{2}d^2y^2+2p^2y^2+\tfrac{1}{2}p^2d^2+p^4=m^2-2my^2,$$
$$\therefore (2m+2p^2-\tfrac{1}{2}d^2)y^2=m^2-\tfrac{1}{2}p^2d^2-\tfrac{1}{16}d^4-p^4,$$

$$\text{Whence } y=\sqrt{\frac{(m^2-\tfrac{1}{2}p^2d^2-\tfrac{1}{16}d^4-p^4)}{2m+2p^2-\tfrac{1}{2}d^2}}.$$

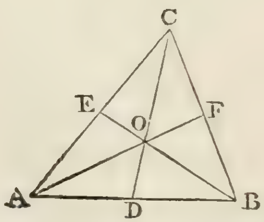
Now $2m=s^2-\tfrac{1}{2}d^2-2p^2=1031512\cdot5$, or $m=515756\cdot25$, which substituted with the other values in the last expression gives $y=\sqrt{\frac{243126056250}{1089000}}=\sqrt{223256\cdot25}=472\cdot5$.

$$\text{Whence } y+\tfrac{1}{2}d=720=\text{AD}$$
$$y-\tfrac{1}{2}d=225=\text{DB}$$

$$\text{Consequently}\begin{cases}\text{AC}=\sqrt{(\text{AD}^2+\text{CD}^2)}=780\\ \text{CB}=\sqrt{(\text{DB}^2+\text{CD}^2)}=375\\ \text{AB}=\text{AD}+\text{DB}=945\end{cases}$$

Which are the three sides required.

Ex. 11. Let ABC be the required triangle, and AF, BE, and CD, the three given lines bisecting the three sides CB, AC, and AB.



Make AF$=a$, BE$=b$, CD$=c$; also BC$=x$, AC$=y$, AB$=z$. It is a well-known property of triangles that

Double the square of a line drawn from any angle of a triangle to the opposite side, together with double the square

of half that side, is equal to the sum of the squares of the other two sides; whence

$$2a^2+\tfrac{1}{2}x^2=y^2+z^2$$
$$2b^2+\tfrac{1}{2}y^2=x^2+z^2$$
$$2c^2+\tfrac{1}{2}z^2=x^2+y^2$$

Or

$$-\tfrac{1}{2}x^2+y^2+z^2=2a^2$$
$$x^2-\tfrac{1}{2}y\ +z^2=2b$$
$$x^2+y^2-\tfrac{1}{2}z^2=2c^2.$$

By addition $\frac{3}{2}(x^2+y^2+z^2)=2(a^2+b^2+c^2)$

or $x^2+y^2+z^2=\frac{4}{3}(a^2+b^2+c^2)$.

Whence, subtracting each of the above equations from this last, we have

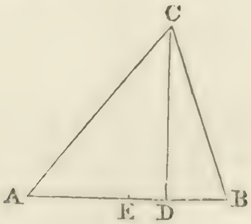
$$\tfrac{3}{2}x^2=\tfrac{4}{3}(a^2+b^2+c^2)-2a^2$$
$$\tfrac{3}{2}y^2=\tfrac{4}{3}(a^2+b^2+c^2)-2b^2$$
$$\tfrac{3}{2}z^2=\tfrac{4}{3}(a^2+b^2+c^2)-2c^2$$
or $x=\sqrt{\frac{8}{9}}(-\frac{1}{2}a^2+b^2+c^2)$
$$y=\sqrt{\tfrac{8}{9}}(a^2-\tfrac{1}{2}b^2+c^2)$$
$$z=\sqrt{\tfrac{8}{9}}(a^2+b^2-\tfrac{1}{2}c^2).$$

Where, by substituting the given values of a, b, and x, viz. $a=18$, $b=24$, $c=30$, we have

$$x=34\cdot176,\ y=28\cdot844,\ \text{and } z=20$$

which are the sides required.

Ex. 12. Let ABC be the triangle, in which AB$=b$, A $-$BC$=c$, and the area $=a^2$.



Draw CD perpendicular to AB, and put AC$=x$, then BC$=x-c$.

$$\therefore a^2=\tfrac{1}{2}b\times \text{CD}$$

and CD$=\dfrac{2a^2}{b}$, the perpendicular, which call p.

Again AD$=\sqrt{\{x^2-p^2\}}$, and BD$=\sqrt{\{(x-c)^2-p^2\}}$.

Hence as AB − AD = BD, we have

$$b-\sqrt{\{x^2-p^2\}}=\sqrt{\{(x-c)^2-p^2\}}.$$

Square both sides, and

$$b^2-2b\sqrt{\{x^2-p^2\}}=c^2-2cx$$

$$\therefore b^2-c^2+2cx=2b\sqrt{\{x^2-p^2\}}.$$

Again, squaring both sides, we have

$$(b^2-c^2)^2+4cx(b^2-c^2)+4c^2x^2=4b^2x^2-4b^2p^2,$$

and by transposition

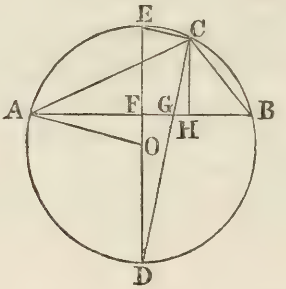
$$4x^2(b^2-c^2)-4cx(b^2-c^2)=4b^2p^2+(b^2-c^2)^2$$

$$\therefore x^2-cx=\frac{b^2p^2}{b^2-c^2}+\frac{1}{4}(b^2-c^2)$$

$$\text{and } x=b\sqrt{\left\{\frac{p^2}{b^2-c^2}+\frac{1}{4}\right\}}+\frac{c}{2}=\text{AC}$$

$$\text{and } x-c=b\sqrt{\left\{\frac{p^2}{b^2-c^2}+\frac{1}{4}\right\}}-\frac{c}{2}=\text{BC}.$$

Ex. 13. Let ABC be the proposed triangle; the base AB $=194=2a$, the diameter ED of the circumscribing circle $=200=2r$, and CG the line bisecting the angle ACB $=66=b$.



Join A with the centre O, and produce CG to D; then because CD bisects the angle ACB, the arc AD = DB, and therefore the diameter ED bisects AB at right angles. Hence OF $=\sqrt{(r^2-a^2)}$

and DF = OD + OF $=r+\sqrt{(r^2-a^2)}=124\cdot310492=c$.

Join CE, and put DG $=x$; then since the two triangles DEC and DGF are similar,

DF : DG :: DC : DE, or

$$c : x :: b+x : 2r$$

$$\therefore x^2+bx=2cr$$

$$\text{and } x = -\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} + 2cr\right)} = 128{\cdot}093446 = d = \text{DG}.$$

Again, $\text{FG} = \sqrt{(\text{DG}^2 - \text{DF}^2)} = \sqrt{(d^2 - c^2)}$; draw CH perpendicular to AB, then by reason of the similar triangles DGF and CGH,

$$\text{DG} : \text{FG} :: \text{CG} : \text{GH}, \text{ or}$$

$$d : \sqrt{(d^2 - c^2)} :: b : \text{GH} = \frac{b}{d}\sqrt{(d^2 - c^2)}$$

$$\therefore \text{FH} = \text{FG} + \text{GH} = \sqrt{(d^2 - c^2)} + \frac{b}{d}\sqrt{(d^2 - c^2)}$$

$$= 30{\cdot}900365 + 15{\cdot}921377$$

$$= 46{\cdot}821742 = e$$

$$\text{AH} = \text{AF} + \text{FH} = a + e = 143{\cdot}821742$$

$$\text{and BH} = \text{BF} - \text{FH} = a - e = 50{\cdot}178258.$$

$$\text{Also DG} : \text{DF} :: \text{CG} : \text{CH}$$

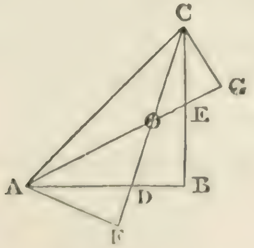
$$\text{or } d : c :: b : \text{CH} = \frac{bc}{d} = 64{\cdot}050837.$$

$$\text{Wherefore AC} = \sqrt{(\text{AH}^2 + \text{CH}^2)} = 157{\cdot}43952$$

$$\text{and BC} = \sqrt{(\text{BH}^2 + \text{CH}^2)} = 81{\cdot}36564,$$

the sides required.

Ex. 14. Let ABC be the proposed triangle; and on the bisecting lines AE, CD, produced, let fall the perpendiculars CG and AF.



Then since (Euc. I. 32,) $\angle$AOF$=$($\angle$ACO$+$ $\angle$CAO)$=45°$, the $\angle$FAO is, also, $=45°$; as are, likewise, for a similar reason, the $\angle$s COG and GCO.

Also, because $\angle$ACO$+$ $\angle$CAO, or $\angle$ACO$+$$\angle$DAO, as well as $\angle$FAO, are each$=45°$, if DAO be taken away, we shall have $\angle$FAD$=$ $\angle$ACF; and for a like reason, $\angle$ECG$=$ $\angle$CAG.

Let, now, AE$=a$, CD$=b$, AF$=$FO$=x$, and CG$=$GO$=xz$; then AO$=\sqrt{2x^2}=x\sqrt{2}$, and CO$=\sqrt{2x^2z^2}=xz\sqrt{2}$; also AG$=x\sqrt{2}+xz$, and CF$=xz\sqrt{2}+x$.

Hence, the triangles CAF, ADF, and CAG, ECG, being similar, we shall have

$$\text{AG}(x\sqrt{2}+xz) : \text{CG}(xz) :: \text{CG}(xz) : \text{EG}=\frac{xz^2}{z+\sqrt{2}}$$

$$\text{CF}(xz\sqrt{2}+x) : \text{AF}(x) :: \text{AF}(x) : \text{DF}=\frac{x}{1+z\sqrt{2}};$$

Therefore AG$-$EG, or $x(z+\sqrt{2})-\dfrac{xz^2}{z+\sqrt{2}}=a$, and

CF$-$DF, or $x(1+z\sqrt{2})-\dfrac{x}{1+z\sqrt{2}}=b$; or

$x(z+\sqrt{2})^2-xz^2=a(z+\sqrt{2})$, and $x(1+z\sqrt{2})^2-x=b(1+z\sqrt{2})$,

From which equations we have

$$x=\frac{a(z+\sqrt{2})}{(z+\sqrt{2})^2-z^2} \text{ or } \frac{a(z+\sqrt{2})}{2(1+z\sqrt{2})},$$

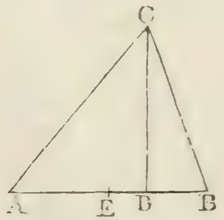
$$x=\frac{b(1+z\sqrt{2})}{(1+z\sqrt{2})^2-1} \text{ or } \frac{b(1+z\sqrt{2})}{2z(z+\sqrt{2})};$$

Whence, by putting these two values of x equal to each other, and then clearing them of fractions, we shall have the cubic equation

$$az^3+2az^2\sqrt{2}+2az=b+2bz\sqrt{2}+2bz^2, \text{ or}$$
$$az^3+2(a\sqrt{2}-b)z^2+2(a-b\sqrt{2})z=b.$$

From which the value of z, and thence that of x may be found; and from these the sides of the triangle, as required.

Ex. 15. Let ABC be the proposed triangle.



Then the base AB$=8$, or AE$=\frac{1}{2}$ base $=4=b$, CD$=4$, $=p$, and AC$+$BC$=12=s$; put ED$=x$.

Then AD $=b+x$, and DB $=b-x$; and consequently

$$\sqrt{\{(b+x)^2+p^2\}}=\text{AC}$$
$$\sqrt{\{(b-x)^2+p^2\}}=\text{BC}.$$

Whence by the question

$$\sqrt{\{(b+x)^2+p^2\}}+\sqrt{\{(b-x)^2+p^2\}}=s.$$

Squaring both sides, and transposing

$$2\sqrt{\{(b+x)^2+p^2\}}\times\sqrt{\{(b-x)^2+p^2\}}=s^2-2b^2-2x^2-2p^2;$$

or, in order to simplify, write $s^2-2b^2-2p^2=2m$

$$\text{then }\sqrt{\{(b+x)^2+p^2\}}\times\sqrt{\{(b-x)^2+p^2\}}=m-x^2.$$

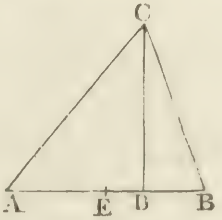
Squaring again, and collecting the terms

$$(b^2-x^2)^2+2p^2(b^2+x^2)+p^4=m^2-2mx^2+x^4$$
$$\text{or } b^4-2b^2x^2+2p^2b^2+2p^2x^2=m^2-2mx^2;$$
$$\text{Therefore } x^2(2p^2-2b^2+2m)=m^2-2p^2b^2-b^4-p^4$$
$$\text{or } x=\sqrt{\left(\frac{m^2-2p^2b^2-b^4-p^4}{2p^2-2b^2+2m}\right)}$$

Whence x, and consequently AD and DB become known; and hence also AC $=\sqrt{(\text{AD}^2+\text{CD}^2)}$, and CB $=\sqrt{(\text{DB}^2+\text{CD}^2)}$, are determined.

In the present case, $p=4$, $b=4$, $s=12$, and $m=40$, whence $x=\frac{6}{5}\sqrt{5}$; therefore AC $=6+\frac{4}{5}\sqrt{5}$, and BC $=6-\frac{4}{5}\sqrt{5}$.

Ex. 16. Let ABC be the proposed triangle.



Then AB $=15=2b$, or AE $=\frac{1}{2}$ base $=7\frac{1}{2}=b$, and $\frac{45}{7\frac{1}{2}}=6$ $=p$, the perpendicular; also the ratio AC : CB :: 3 : 2, or $m:n$, and ED $=x$.

Then, as in the preceding example,

$$\sqrt{\{(b+x)^2+p^2\}}=\text{AC}$$
$$\sqrt{\{(b-x)^2+p^2\}}=\text{CB},$$

And by the question

$$\sqrt{\{(b+x)^2+p^2\}} : \sqrt{\{(b-x)^2+p^2\}} :: m : n, \text{ or}$$
$$\{(b+x)^2+p^2\} : \{(b-x)^2+p^2\} :: m^2 : n^2,$$

Whence

$$n^2b^2+2n^2bx+n^2x^2+n^2p^2=m^2b^2-2m^2bx+m^2x^2+m^2p^2$$
$$(m^2-n^2)x^2-2b(m^2+n^2)x=(n^2-m^2)b^2+(n^2-m^2)p^2$$

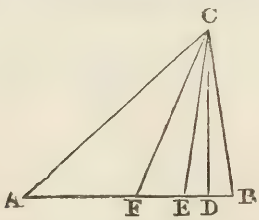
$$\text{Or } x^2-2b\left(\frac{m^2+n^2}{m^2-n^2}\right)x=-b^2-p^2$$

$$\text{Whence } x=\left(\frac{m^2+n^2}{m^2-n^2}\right)\pm\sqrt{\left\{b^2\frac{(m^2+n^2)^2}{(m^2-n^2)^2}-b^2-p^2\right\}}$$

So that the question evidently admits of two cases: one when the perpendicular falls upon the base, and the other when it meets the extension of the base. As the lower sign is adapted to the former case, represented by the figure, we shall make use of it in the numerical calculation.

By substituting the values of b, p, m, and n, we have $x=2\cdot529437$, AC$=\sqrt{\{(b+x)^2+p^2\}}=11\cdot68716$, and BC$=\sqrt{\{(b-x)^2+p^2\}}=7\cdot79144$.

Ex. 17. Let ABC be the proposed triangle, and let the perpendicular CD$=24=p$, CE the line bisecting the angle ACB$=25=b$, and CF, the line bisecting the base, $=40=c$.



Then (Euc. I. 47) ED$=\sqrt{(\text{CE}^2-\text{CD}^2)}=7=m$,

Also FD$=\sqrt{(\text{FC}^2-\text{CD}^2)}=32=n$;

And EF$=$FD$-$ED$=25=q$.

Also, let half the base AF$=$FB$=x$; then

AE$=x+q$, EB$=x-q$; AD$=x+n$, DB$=x-n$;

$$\text{Hence}\quad \text{AC}=\sqrt{\{(x+n)^2+p^2\}}$$
$$\text{BC}=\sqrt{\{(x-n)^2+p^2\}}$$

And from (Euc. VI. 3) we have

AC : BC :: AE : EB, or

$$\sqrt{\{(x+n)^2+p^2\}} : \sqrt{\{(x-n)^2+p^2\}} :: x+q : x-q;$$

Whence

$$\{(x+n)^2+p^2\} \times (x-q)^2 = \{(x-n)^2+p^2\} \times (x+q)^2,$$

Which, by multiplying, cancelling, &c. becomes

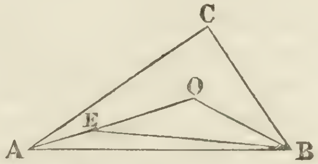
$$nx(x^2+q^2) = qx(x^2+n^2+p^2),$$

$$\text{Where } x^2 = \frac{qn^2+qp^2-nq^2}{n-q}, \text{ or}$$

$$2x = 2\sqrt{\frac{q(n^2+p^2)-nq^2}{n-q}}.$$

the base of the triangle; which, by substituting the numeral values of q, n, and p, gives $\frac{200}{7}\sqrt{14}$; from which and the given lines, the other two sides are readily obtained.

Ex. 18. Let ABC be the required right angled triangle, and O the centre of the inscribed circle.



In OA take OE=OB; then AE is equal to the given difference. Join EB, and we shall have 2∠OEB=OEB+OBE= OAB+EBA+OBE=OAB+OBA=$\frac{1}{2}$(BAC+ABC) = 45°; hence ∠OEB=OBE=22° 30′, and ∠AEB=180°−OEB=180°−22° 30′=157° 30′ Hence we have AB=10, AE=2, to find the ∠ABE. The computation is as follows:

As AB	10	ar. comp.	9·0000000
: sin. ∠AEB . .	157° 30′	. . . .	9·5828397
:: AE	2	. . . .	0·3010300
: sin. ∠ABE . .	4° 23′ 22″·3	.	8·8838697

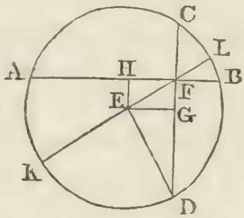
Hence ∠ABO=ABE+OBE=4° 23′ 22″·3+22° 30′=26° 53′ 22″·3, and ∠ABC=2∠ABO=53° 46′ 44″·6.

Again AC = AB × sin. ∠ABC, and BC = AB × cos. ∠ABC

∠ABC	53° 46′ 44″·6	sin.	9·9067359
AB	10 . . .		1·0000000
	AC = 8·067442		0·9067359

∠ABC	cos.	9·7715145
AB		1·0000000
BC = 5·909007		0·7715145

Ex. 19. Let ABC be the required circle, AB $=a$ and DC $=b$ the two given lines, KL the required diameter, and EF $=c$ the given distance.



Join ED and draw the two perpendiculars EH, EG; also call FG $=x$, and HF or EG $=y$; then

$$x^2+y^2=c^2 \text{ (Euc. I. 47.)}$$

$$(\tfrac{1}{2}a+y)(\tfrac{1}{2}a-y)=(\tfrac{1}{2}b+x)(\tfrac{1}{2}b-x). \text{ (Euc. III. 35.)}$$

The latter of which gives $x^2-y^2=\frac{1}{4}b^2-\frac{1}{4}a^2$,

and, as before, $x^2+y^2=c^2$,

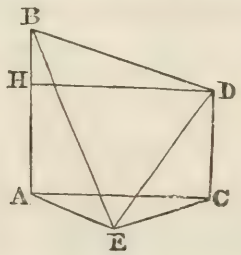
then by subtraction $y^2=\frac{1}{8}a^2-\frac{1}{8}b^2+c^2=\text{EG}^2$.

Again $\text{ED}^2=\text{EG}^2+\text{GD}^2$, or

$$\text{ED}=\sqrt{(\tfrac{1}{8}a^2+\tfrac{1}{8}b^2+\tfrac{1}{2}c^2)}=\text{radius, or}$$

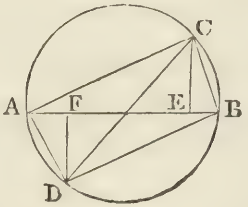
$$\text{KL}=\sqrt{\tfrac{1}{2}a^2+\tfrac{1}{2}b^2+2c^2)}=\text{diameter.}$$

Ex. 20. This question, of which the figure is as follows, does not require the assistance of algebra.



For $BD=\sqrt{(HB^2+HD^2)}=\sqrt{(400+14400)}=\sqrt{14800}$,
whence ED and BE are each $=\sqrt{14800}$;
consequently $EC=\sqrt{(ED^2-DC^2)}=\sqrt{(14800-6400)}=$
$\sqrt{8400}=20\sqrt{21}$; and $AE=\sqrt{(BE^2-AB^2)}=$
$\sqrt{(14800-10000)}=\sqrt{4800}=40\sqrt{3}$, as required.

Ex. 21. Let ACBD be the given trapezium,
Where $AD=6$, $DB=4$, $CB=5$, and $CA=3$.



Then draw the diagonal AB, and let fall upon it the two perpendiculars CE and DF, and make $CE=p$ and $DF=p'$; also put the required diameter $=x$. Then by (Euc. VI. C.)

$$p\,x=5\times3=15$$
$$p'\,x=4\times6=24, \text{ or}$$
$$x(p+p')=39.$$

In the same manner calling q and q' two perpendiculars falling on the diagonal CD, we have

$$q\,x=3\times6=18$$
$$q'x=4\times5=20, \text{ or}$$
$$x(q+q')\ =38,$$

Whence $(p+p') : (q+q') :: 39 : 38$.

But the products of these perpendiculars and their respective diagonals are equal to double the area of the trapezium; whence AB : CD :: 38 : 39.

Also (Euc. VI. D,) $AB\times CD=AC\times BD+AD\times CB$;

Let therefore $AB=y$, and $CD=z$; then

$$y : z :: 38 : 39$$
$$\text{and } yz=42;$$
$$\text{whence } y=\sqrt{\frac{42\times38}{39}}, \text{ and } z=\sqrt{\frac{42\times39}{38}}.$$

Hence knowing the three sides of the triangle ABC, *viz.* $AC=3$, $BC=5$, and $AB=\sqrt{\frac{42\times38}{39}}=2\sqrt{\frac{133}{13}}$, the perpendicular may be found thus.

By Euc. (II. 13,) $2AB.AE = AB^2 + AC^2 - BC^2$

$$\therefore\ 4AE\sqrt{\frac{133}{13}} = \frac{4\times 133}{13} - 16$$

$$\text{or}\quad AE\sqrt{\frac{133}{13}} = \frac{81}{13}$$

$$\therefore\ AE = \sqrt{\frac{81^2}{133\times 13}} = \sqrt{\frac{6561}{1729}},$$

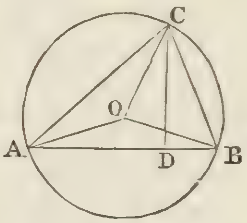
$$\text{and } CE = \sqrt{(AC^2 - AE^2)} = \sqrt{\frac{9000}{1729}}.$$

$\therefore$ (by Euc. VI. C) the diameter $= AC.BC \div CE =$

$$15\sqrt{\frac{1729}{9000}} = 15\sqrt{\frac{17290}{90000}} = \frac{1}{20}\sqrt{17290} = 6 \cdot 57457.$$

Ex. 22. This question, of which the figure is as follows, is nothing more than having the three sides of a triangle to find the radius of the circumscribing circle.

Let therefore ABC be the given triangle, of which



$AC = 25 = a$, $AB = 30 = b$, $CB = 20 = c$, and O the required centre. From which the perpendicular CD is readily found.

For, by putting $AD = x$, and $DB = y$, we have

$$x^2 - y^2 = a^2 - c^2$$

$$\text{and } x + y = b,$$

$$\text{Whence } x - y = \frac{a^2 - c^2}{b}.$$

$$\text{By addition } x = \frac{b^2 + a^2 - c^2}{2b},$$

And hence

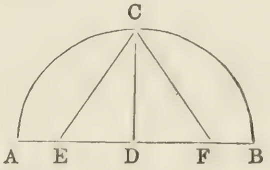
$$CD = \sqrt{\left\{a^2 - \frac{(b^2 + a^2 - c^2)^2}{4b^2}\right\}} = \frac{25}{4}\sqrt{7}.$$

Again (Euc. VI. C.) Diam. $\times CD = AC \times CB$,

$$\text{Whence Diam.} = \frac{25 \times 20}{\frac{25}{4}\sqrt{7}} = \frac{80}{\sqrt{7}} = \frac{80}{7}\sqrt{7} =$$

30·237158; and consequently OA, OB, or OC $=15 \cdot 118579$, the distance sought.

Ex. 23. Let ABC be the proposed semicircle, ECF the given equilateral triangle, whose area $=100=a$, and whose perpendicular CD is equal to half the required diameter.



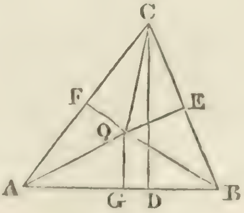
Let CE, the side of the triangle, $=x$; then ED $=\frac{1}{2}x$, and CD$=\sqrt{(x^2-\frac{1}{4}x^2)}=\frac{1}{2}x\sqrt{3}$.

But DC $\times$ ED$=\frac{1}{2}x \times \frac{1}{2}x\sqrt{3}=100$, the area.

$$\text{Whence } \tfrac{1}{4}x^2\sqrt{3}=100, \text{ or } x=\frac{20\sqrt[4]{3}}{\sqrt{3}}$$

Consequently AB$=2$CD$=x\sqrt{3}=20\sqrt[4]{3}$, the diameter required.

Ex. 24. The rectangle of the perpendicular and diameter of the circumscribing circle of any triangle, is equal to the rectangle of the two sides. (Euc. VI. C)



Whence the present question reduces to this, *i.e.* given the perpendicular $=p$, the radius of the inscribed circle OG, OE or OF$=r$, and the product of the two sides $=$Rp, to find the sides.

Let, therefore, the segment AD$=z+x$, and DB$=z-x$.

Then (Euc. I. 47.)

$AC=\sqrt{\{(z+x)^2+p^2\}}$

$BC=\sqrt{\{(z-x)^2+p^2\}}$

$AB=2z$.

Also because $AB\times CD=(AB+BC+AC)\times OG$, we have

$\sqrt{\{(z+x)^2+p^2\}}\times\sqrt{\{(z-x)^2+p^2\}}=2Rp$, and

$\sqrt{\{(z+x)^2+p^2\}}+\sqrt{\{(z-x)^2+p^2\}}\times r=2(p-r)z$;

Whence, squaring the latter equation, and substituting for the double rectangle, we have

$$2z^2+2x^2+2p^2+4Rp=\frac{4(p-r)^2z^2}{r^2},$$

$$\text{or } x^2=\frac{2(p-r)^2}{r^2}z^2-z^2-p^2-2Rp$$

or by putting $\frac{2(p-r)^2}{r^2}-1=m$, and $p^2+2Rp=n$

$$x^2=mz^2-n.$$

Also, squaring the first equation, and multiplying,

$$(z^2-x^2)^2+2p^2(z^2+x^2)+p^4=4R^2p^2;$$

in which, substituting for x^2 the value found above, we have

$$(z^2-mz^2+n)^2+2p^2(z^2+mz^2-n)=4R^2p^2-p^4;$$

that is,

$$z^4(1-m)^2+2n(1-m)z^2+2p^2(1+m)z^2+n^2-2p^2n$$
$$=4R^2p^2-p^4;$$

which equation reduces to

$$z^4+\frac{2n(1-m)+2p^2(1+m)}{(1-m)^2}z^2=\frac{4R^2p^2-p^4-n^2+2p^2n}{(1-m)^2},$$

where by re-establishing the value of n in the second side, it becomes $=0$, and there remains

$$z^2=\frac{2n(m-1)-2p^2(m+1)}{(m-1)^2},$$

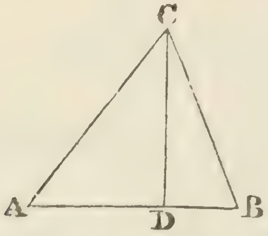
Or, by substituting for m and n their respective values, we have

$$z=\frac{r\sqrt{(2Rp-4Rr-r^2)}}{p-2r},$$

$$\text{Whence } 2z=\frac{2r\sqrt{(2Rp-4Rr-r^2)}}{p-2r}$$

the base; from which the other sides may be determined.

Ex. 25. Let ABC be the proposed triangle, whose base $AB=2a$, and perpendicular $CD=a$.



Suppose $x=\frac{1}{2}(AC+BC)$, and $y=\frac{1}{2}(AC-BC)$;
then $x+y=AC$, and $x-y=BC$.

Also, by the nature of the problem $AC^3+BC^3=3AB^3$,

$$\text{That is } (x+y)^3+(x-y)^3=24a^3$$
$$\text{or } 2x^3+6xy^2=24a^3,$$
$$\therefore y^2=\frac{12a^3-x^3}{3x}.$$

Now the perimeter of the triangle $=2x+2a$,

$\therefore \frac{1}{2}$ the perimeter $=x+a$;

Hence $\frac{1}{2}$ perim. $-AC=(x+a)-(x+y)=a-y$,
$\frac{1}{2}$ perim. $-BC=(x+a)-(x-y)=a+y$,
$\frac{1}{2}$ perim. $-AB=\ x+a\ -2a\ \ =x-a$.

$\therefore$ the area of the triangle

$$=\sqrt{\{(x+a).(a-y).(a+y).(x-a)\}}$$
$$=\sqrt{\{(x^2-a^2).(a^2-y^2)\}};$$

and the said area is also $=\frac{1}{2}(AB.CD)=a^2$;

$$\therefore (x^2-a^2).(a^2-y^2)=a^4,$$
$$\text{which gives } y^2=\frac{a^2(x^2-2a^2)}{x^2-a^2}.$$

$$\text{But it has been shown that } y^2=\frac{12a^3-x^3}{3x}$$
$$\therefore \frac{a^2(x^2-2a^2)}{x^2-a^2}=\frac{12a^3-x^3}{3x}$$

which reduces to

$$x^5+2a^2x^3-12a^3x^2-6a^4x+12a^5=0.$$

The best method of finding an integral root of an equation of this kind, is to substitute for x the various divisors of the last term $12a^5$, which will, if the equation admits of an integral root, be always successful.

In this equation one value of x is $2a$.

$$\text{Hence } y^2=\frac{12a^3-8a^3}{6a}=\frac{2}{3}a^2.$$

$$\therefore\ y=a\sqrt{\frac{2}{3}}=\frac{a}{3}\sqrt{6}$$

$$\text{AC}=x+y=2a+\frac{a}{3}\sqrt{6}=a(2+\frac{1}{3}\sqrt{6})$$

$$\text{and BC}=x-y=2a-\frac{a}{3}\sqrt{6}=a(2-\frac{1}{3}\sqrt{6}).$$

THE END.

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